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ABSTRACT

This is a book of readings from the "Arithmetic Teacher" on selected topics in geometry. The articles chosen are samples of material published in the journal from its beginning in February 1954 through February 1970. The articles are of three major types. The first is classified "involvement." These articles describe geometry units in which the students build geometrical models, play games, and draw geometrical objects. Another article in this classification focuses on a teacher preparation course in which the future teachers experience the learning activities of the students. The second group of articles is categorized "instruction-techniques." These articles focus on the techniques of teaching units in informal geometry using mirrors, models, toys, and Mobius bands. The third type of article is termed "instruction-rationale." This type of article gives reasons why geometry should be taught in the elementary grades and tells what parts of geometry should be taught. Included in the book is a bibliography of articles published in the "Arithmetic Teacher" pertinent to geometry. (Author/CT)

Readings in Geometry

from the *Arithmetic Teacher*

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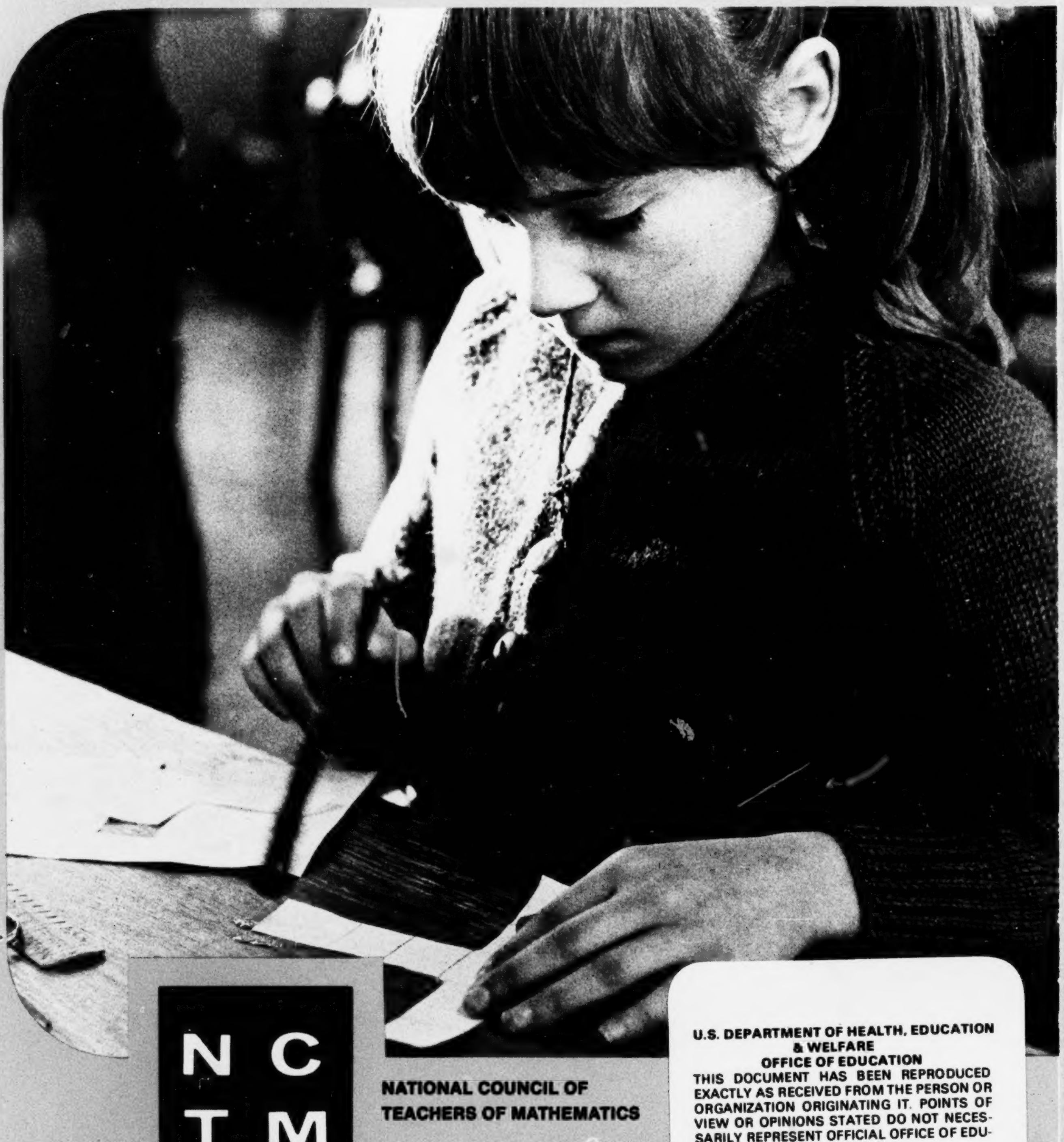

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Readings in Geometry from the *Arithmetic Teacher*

edited by

MARGUERITE BRYDEGAARD

and

JAMES E. INSKEEP, JR.

National Council of Teachers of Mathematics
Washington, D.C.

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Introduction

This is a book of readings from the *Arithmetic Teacher* on selected topics in geometry. The articles reprinted here are samples of material published in the journal from its beginning in February 1954 through February 1970.

The selection of readings to represent the contribution of the *Arithmetic Teacher* to geometry for grades K–8 was not easy. In some cases an article was chosen because it develops a topic or idea that is unique and original or because it covers an area for which few articles have been written. When one article paralleled or overlapped another, a choice was made on the basis of an evaluation by the panel of reviewers. Since the number of pages was limited in order to provide an inexpensive publication, articles on the topics of research and measurement are not included in the readings but are represented in the Bibliography. A rather complete list of these and other pertinent articles is contained in the Bibliography, in which the latest citation is from the issue of May 1970. This listing includes the articles reprinted here.

In recent years the *Arithmetic Teacher* has carried a variety of articles dealing with geometry in grades K–8. It is interesting to note the growth of attention given to the topic. Before 1959 no articles in the journal dealt specifically with geometry. Even after that time the flow of published articles was minuscule until the middle sixties, when the increase in publication was dramatic. As an indication of the increase, note that the following six issues emphasized the theme of geometry: October 1966, February 1967, October 1967, December 1968, October 1969, and February 1970.

The increase in publication developed in response to a new concept of geometry for the elementary school. Geometry has classically been associated with Euclid and an axiomatic approach to study, and from this viewpoint geometry was not considered appropriate for the elementary school program. The new approach is neither axiomatic nor narrowly defined but rather places emphasis upon such ideas as the environment, informal approaches, and readiness.

The term "*ge*ometry," from its Greek derivation, has the meaning "*earth* measurement." Today our concern is with the exploration, study, and measurement of *space;* and — to coin a word — "*space*ometry" would more aptly identify the subject. The broader concept of space

7

relationships is a significant part of the program for grades K–8. It is realized that each learner intuitively and naturally senses and responds to his spatial environment. Ideas such as position, direction, distance, size, and shape are sensed in the child's earliest experiences. It appears to be true that by the time a child enters school, his knowledge and understanding of geometry may be greater than his development of concepts of number.

As published information about the teaching of geometry in grades K–8 has increased, so has the need to disseminate it. With evidence of demand and need, this book of readings has been assembled to promote wider distribution of quality material dealing with this subject.

1

Involvement

The articles in this publication have been grouped according to two major categories — *involvement* and *instruction*. Those falling under the second category have again been grouped according to whether the emphasis is on *technique* of instruction or its *rationale*. While no article is strictly limited to any one of these categories, it is thought that the selected articles highlight either involvement or the techniques or rationale of instruction.

Involvement! Personal involvement and learning are inseparable. One of the basic approaches to the mathematical involvement of children is through the study of geometry. When studied intuitively, the environment of a child opens a wide vista of geometric ideas. This environmental approach to geometry lays a foundation of readiness for informal and formal geometry at a later age.

Involvement of the learner! In the beginning article we see Joey, age six, involved with geometric construction and problem solving that lead him to significant mathematical ideas. His inventions were guided by the "teacher" who provided the materials and the freedom to do problem solving and also by his sister, Maureen, who assisted him.

Involvement of a classroom of learners and their teacher is described in Black's article. As its title suggests, geometry *is* alive! The author presents many ideas that help teachers to identify geometric concepts and to sense how to organize a laboratory for experimental involvement. MacLean, in the article that follows and complements Black's article, describes the K–6 Experimental Project of the Ontario Mathematics Commission. The eight activities that are described illustrate ideas a teacher can adapt for his own classroom.

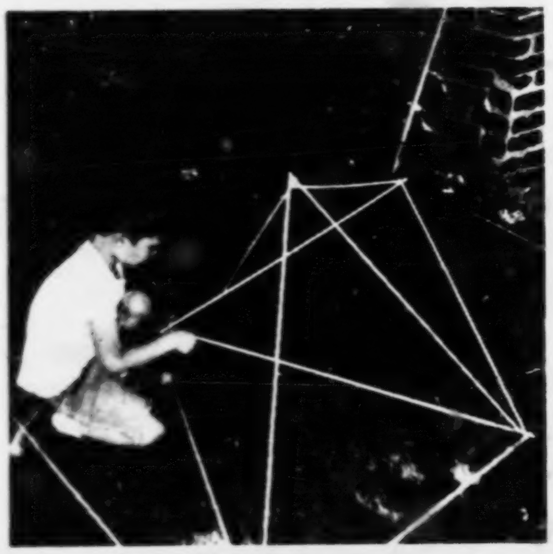
Involvement of preservice teachers is discussed by Kipps in describing her experimental course for university students. Using a laboratory approach, she challenged her students to experience some of the things they should practice in their classrooms with children.

These four articles illustrate ideas of involvement of children and teachers. The message is simple and exciting. He who pursues involvement as a basic procedure for learning will not only search for techniques that are productive but will seek effective rationale for the concepts that he teaches.

3 - 4

With sticks and rubber bands

JOSEPH SCOTT
North Valley Stream, New York

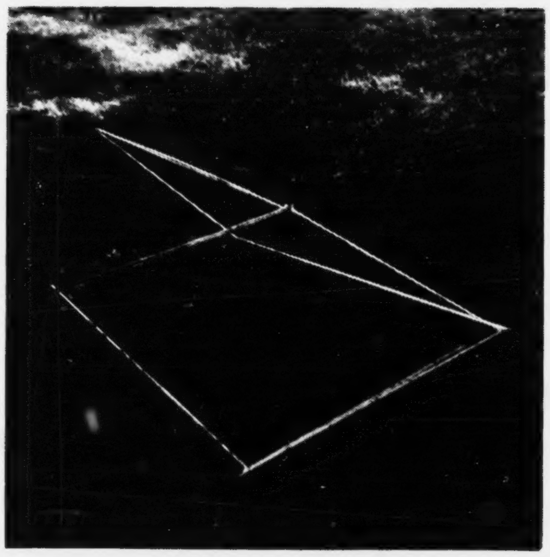


This is The first invention that I did.
This was easy.

I had a good time making Things.

This is the monster from outer Space.

10

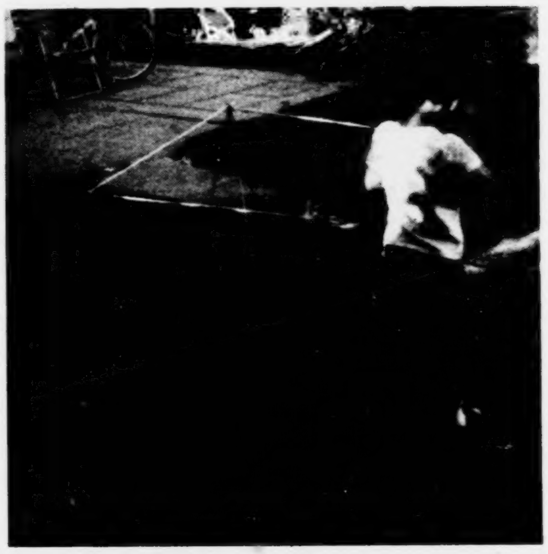


this is the mouth
of a duck.
I Think one Time
I Stuck somebody
in It, but they got
out an easy way.

This is The building That I was building. My Sister Maureen was holding one of the corners for me. I finished the building but it wouldn't

stand up by itself
because it didn't have
any triangle in it.

This is the nose cone.



This is the bottom of
the rocket.

I finished the rocket.

* * *

Joey, who is six years old, spent about seven hours with this project, about two hours at a time.

The sticks are 1/4" dowel rods, three

feet long, and available at lumber yards at about $.08 apiece. About 30 sticks are sufficient. Regular rubber bands were used.

What learning was involved?

Mathematics:	two-dimensional shapes (triangle, square, pentagon, and others). three-dimensional shapes (tetrahedron, octahedron, cube, and others). symmetry of figures, counting edges, corners, faces.
Physics:	rigidity of the triangle as used in bridges, buildings.
Art:	positioning shapes, symmetry (one could cover these shapes with crepe paper).
Language Arts:	Joey wrote about what he did and read his story.
Problem Solving:	problem solving that involved concentration and freedom.

—DON COHEN, Madison Project Resident Coordinator, New York, New York.

Geometry alive in primary classrooms

JANET M. BLACK *Barrie, Ontario, Canada*

Janet M. Black is a classroom teacher in Barrie. She has experimented extensively in teaching geometry to children. The typical lesson that she describes was reported at The National Council of Teachers of Mathematics meeting in Calgary, August 1966.

Planning an experimental course in geometry for forty-two Grade 3 students was quite a challenge. Five phases of geometry were to be included:

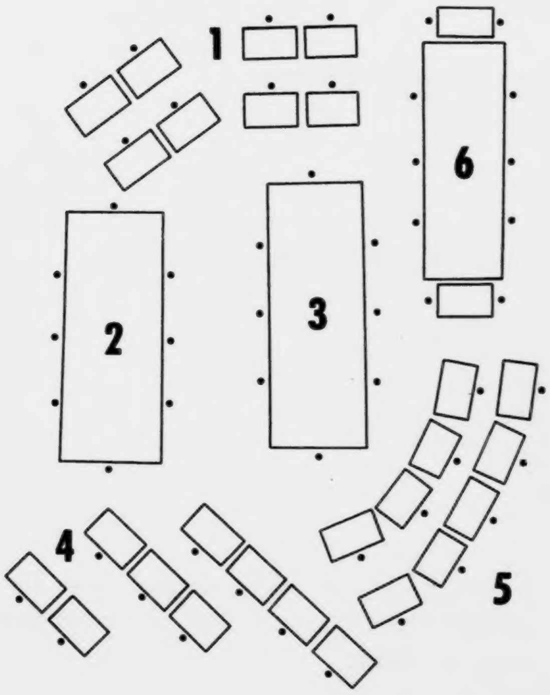
1. A study of solids and their basic properties to develop in children an awareness of shape in connection with their environment
2. A study of plane figures through the examination of the surfaces of solids and of real objects around school and home
3. A study of lines as the edges of solids and of points as fixed locations in space through construction of models and experimentation with as many concrete objects as possible
4. A review of concepts through the "building" of solid shapes from flat surfaces and the construction of models in the environment
5. Symmetry, tiling, and nesting involving solids and flats

During geometry periods, work stations were set up. The children were challenged orally by assignment cards, by the tape recorder, or by charts to examine certain materials to see what could be discovered. The following description of a typical lesson involving geometric concepts took place at about stage two of the five phases of geometry. Children were in a transitional stage proceeding from the study of solids to the study of plane shapes, and the activities were designed with three definite ideas in mind:

1. To make the transition from the study of solids to the study of plane shapes as natural for the children as possible
2. To establish through actual experimentation with physical materials an intuitive understanding of both solid and plane shapes
3. To provide for primary children the time to discover and the freedom to experiment within the classroom at each individual's own chosen speed

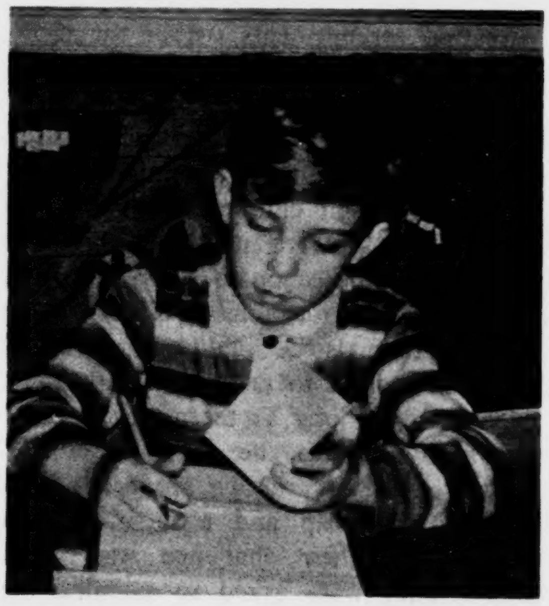
Seating arrangements within the classroom were changed frequently, depending upon the number of work stations required for each lesson. For this particular session six stations which could accommodate approximately eight children each were set up with the required materials. The stations are numbered on the following floor plan but were not numbered during the actual lesson, since children proceeded from one station to another in any order they chose as long as there was a place free at the chosen site.

Children at centre one are using half-sheets of foolscap and a large wooden solid each. Their challenge was given orally. They were asked to examine the solid on the desk, and on the paper list all the things that they could think of



which have the same shape as the solid. For instance, the boy in the picture is examining the cube, and he may list such things as the following:

1. My sister's jack-in-the-box
2. The blocks I have at home
3. The plastic pencil holder
4. Dice
5. The box on the green shelf



CENTRE ONE

When he is finished he is encouraged to discuss his list with the person next to him. He may then move to another desk in the same section and prepare a similar list for a different solid, or he may proceed to a new work centre—whichever he chooses.

CENTRE TWO

Children working at this section have a large chart to guide their examination of solids. Since the transition from three-dimensional shape to two-dimensional shape is a general aim for this lesson, the chart contains questions such as these:

1. List the shapes which look like boxes.
2. Which solids have flat surfaces shaped like circles?
3. Find all the solids which have more than five flat surfaces.
4. Which rolls more easily?
   a) a solid with two flat surfaces
   b) a solid with five flat surfaces
5. Spin the square pyramid. What other solid does it look like while it is spinning?

Children are encouraged actually to handle and experiment with the solids as they prepare their charts.

The third centre deals with the problem of rigidity. Children have examined a large cube and have built its "skeleton"

Centre Three

from straws and pipe cleaners. Their challenge was also given orally. They were to find out whether or not the skeleton of the cube would stand upright and hold its shape. If it wouldn't, the children were to discover a way to make the structure rigid by adding more straws.

Children were encouraged to work in twos or threes at this centre so that they could discuss several ideas and try them out before deciding which one worked best. This centre was, by the way, a real study in "citizenship" as well as geometry, since a discussion involving ideas is not always a placid one, and children sometimes need help to work effectively in a situation such as this.

The day before this lesson, children had traced the faces of the large wooden solids onto sheets of paper and reproduced the shapes of the faces on geo-boards.

Centre Four

At this centre nine individual work cards have been prepared, all dealing with the shapes of faces (triangles, rectangles, and squares). In this particular instance, the fellow in the picture is working through an assignment dealing with the positioning of one face shape over another—an intuitive experience with intersection of lines, but directly mentioning only the shape of a three-sided flat surface. Since each assignment card at this station is different, the boy may choose to continue to work in this section at a different desk with a different card or to proceed to any other station which interests him upon completion of his present assignment. In many cases it is possible to put the answers to assignment cards on the back of the card, and the student therefore is capable of marking his own work.

Centre Five

At this centre children worked individually with an assignment card as a guide. The instructions involved choosing a certain number of straws in order to build a shape that would stand by itself. The boy in the picture has experimented with six straws, joining them at the vertices with pipe cleaners. He was, of course, amazed at the resulting tetrahedron, which he called a "four-sided pyramid". Children are encouraged to invent their own names for many shapes, since in some cases, particularly in Grade 1, the technical name

CENTRE SIX

is much too difficult for many of the children.

There are many games available which will help children progress in their understanding of spatial relationships. The game included in this lesson was originally known as "Kahla", an Egyptian game sometimes played on folded paper with popcorn for counters. On large cardboard sheets with thick plastic discs for counters, our version of Kahla, known as "Calabash", encourages estimation visually and mentally. The children follow the rules of the game and attempt to win all or most of the calabash counters from a partner. The players then record their scores on the blackboard and move on to another division.

Following are other examples of games which develop geometry concepts:

1. Contack
2. Hex
3. Three-dimensional tic-tac-toe (Cubic)
4. Parquetry blocks involved in games
5. Chinese Checkers

CULMINATION

At the end of the lesson period, various reports are given orally by children working at stations 3, 4, and 5. The chart from centre two is developed co-operatively by individual members of the class while others mark their work (anyone who did not reach the centre picks up an extra chart, and works along orally with the rest of the class, filling in answers if he wishes to do so). The lists from centre one are read aloud by individuals and pinned around the "Shapes Table" in a corner of the classroom. The blackboard containing the individual scores from Calabash is examined, and a discussion follows concerning the person who won by the highest score, combinations of different scores, and the possibility of recording a "score in the hole" for some people.

The quest for an improved curriculum*

J. R. MacLEAN

Assistant Superintendent, Curriculum Division
Ontario Department of Education

Editor's Note.—Mr. MacLean was one of the speakers at the Calgary Meeting of the National Council of Teachers of Mathematics in August 1966. His article complements that of Janet Black in this issue of the journal, "Geometry Alive in Primary Classrooms."—Marguerite Brydegaard.

For the past several years teachers have been almost overwhelmed by successive waves of propaganda that has generally been critical of our "traditional" mathematics programs. Unless set notation and symbolism, non-decimal numeration, modular arithmetic and strict attention to correct terminology (such as distinction between number and numeral) were included in our courses of study, we were not up-to-date. Our students would be ill-prepared for their responsibilities in the Space Age and we were effectively sabotaging the future of our country.

It is unfortunate that this "new math" concept has confused as much as it has changed elementary school mathematics. There are many reasons why this has occurred, but the most significant are these: the haste with which the new ideas were applied; a lack of understanding on the part of some of the mathematicians designing the programs about how children learn and what they need to learn; the general unpreparedness of teachers and supervisors which has fostered the idea that the "new" completely replaces the "old" rather than clarifying, supplementing and providing more meaningful approaches; and finally, the lack of unbiased evaluation of the newer materials and techniques.

The K-6 Kingston Study Group

Conscious of these weaknesses and yet alert to the necessity of applying new knowledge, both in mathematics and in pedagogy, to the development of new and better programs in elementary school mathematics, the Ontario Mathematics Commission, with the financial support of the Ontario Curriculum Institute, undertook the task of exploring and collating the various experiments and approaches that are being carried out in Ontario and elsewhere. The Commission appointed ten teachers to a committee—eight directly involved in elementary schools, one from the high schools and one from the universities—and asked them to suggest possible revisions of the present curriculum and to make recommendations for the implementation of the suggested changes.

The K-6 Mathematics Study Group met

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during the summer of 1965 at the Royal Military College in Kingston, and produced a report which was subsequently published by the Ontario Curriculum Institute. This report has been described as one of the most significant documents to appear on the educational scene in many years. All teachers who are concerned with the teaching of mathematics are urged to read this report. Some of the recommendations of the committee have been included in the Interim Revision in Mathematics recently published by the Ontario Department of Education.

The K-6 Experimental Project

The proposals in the report for implementing a new program have been considered by the Commission, especially item 7—"that a full time group be established . . . to prepare teachers' handbooks and other materials for selected topics in specific grades so that extensive experimental work can be carried out in the academic year 1966-67." Once more the Curriculum Institute agreed to finance a project that would involve many teachers and students throughout the province. It was decided to prepare materials for the study of geometry in Grades 1, 2, and 3, and Graphs, Mappings and Relations in Grades 4, 5, and 6. Selected teachers from all sizes of school jurisdictions met in Toronto during July for training sessions in presenting the materials.

Perhaps the most vital aspect of this experimental program is the change in pedagogical techniques which in turn transform the classroom from the usual array of students sitting in desks arranged in neat rows to a virtual "laboratory for learning." Pupils are actively engaged in solving problems with physical materials that they can manipulate themselves. This approach is illustrated by an ancient Chinese proverb—

> "I hear, and I forget;
> I see, and I remember;
> I do, and I understand."

Many of the ideas and techniques presented in the Nuffield Mathematics Teaching Project publications have been adapted for use in the experimenting classrooms. The writing team was greatly encouraged through the knowledge that other people in other countries were working with ideas not too divergent from its own.

The following sample lesson or more appropriately "experience description" has been taken from the outline prepared by the writing team and presented to the teachers attending the summer course. Running through it is the central notion that children must be set free to make their own discoveries and think for themselves.

Lesson Topic: *Examining Faces*

This lesson was presented after three previous periods spent examining, discussing and manipulating solids. Each student picks a partner to work with during the lesson period. The partners are given a piece of paper and crayons, and each pupil chooses a solid from the set provided. Their assignment is to trace the flat surface of each solid onto a sheet of paper, attempt to select shapes that are different, and then discuss why they are different.

The class might choose a circular face to examine. One member is appointed secretary-chairman. Each pupil should draw a ring around the circular faces they have traced and suggest things that make the circle different from the other shapes. The secretary records all suggestions, and the class then evaluates the chart he produces. Other faces should be examined in a similar manner before students proceed to the activity centres, indicated in Figure 1 (next page).

(Activity centres are suggested as one possible way of providing for experimentation-thinking-communication, which seems to be an obvious line of development for a child.)

The students proceed from one activity to another, working through assignment cards, chart activities, logic games and measurement experiences. When they finish an activity they are allowed to move to

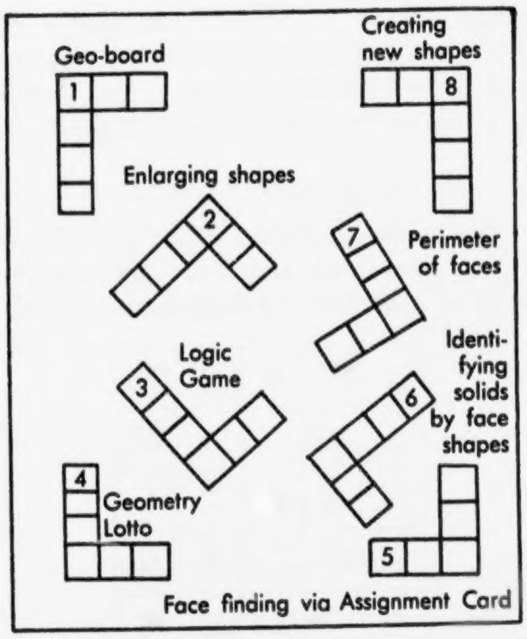


FIG. 1.—A suggested classroom arrangement showing activity centres.

any other centre provided there is an empty space. The teacher moves about the room, guiding, encouraging and helping when requested. In this classroom atmosphere the teacher's role is not to stop the children talking, but rather to ensure that there is something worthwhile for them to talk about. There is a place for lively discussion, and the quality of the discussion will be directly dependent upon the quality of the class-teacher relationship.

Do not expect any miraculous change in the behaviour of children immediately after introducing these experience-centred classroom groupings. Initially it may be better to try fewer groups and allow the pupils time to adjust to the freer atmosphere and to develop respect for the opinions of others. This latter attribute follows smoothly the respect the teacher shows for the student's point of view. Remember, real discussion, wherever it appears, is provoked by experience. The situation supplies the starting point; the discussion that ensues should widen the children's horizons and open up many new avenues of exploration.

Here are brief descriptions of the eight activities.

1. GEO-BOARD

Geo-boards are square (in this case 12″ by 12″) boards with nails driven in at 1″ intervals. Assignment cards could be similar to the following:

Assignment:

1. On your geo-board make all the faces of the triangular prism.
2. Two of the faces are alike. What shape do they have?
3. Three faces are alike. What shape do they have?
4. Count the number of sides on one triangle.
5. How many faces do you have on a rectangle?

Check your answers on the back of the card.

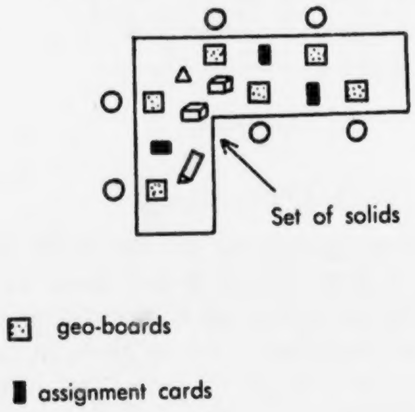


FIGURE 2

Students may use elastic to recreate the shapes of faces, answer the questions either orally or on paper, then check their answers.

2. ENLARGING SHAPES

Again students work from assignment cards. They create designs using known shapes and enlarge the same design on graph paper.

Assignment Card Samples

1. Use graph paper to design a house—use a triangle and a rectangle.

2. By making the sides longer draw another house. It will be the same shape, but it will be larger in size.
3. How much bigger do you think the new shape is than the first shape?

Possible Results

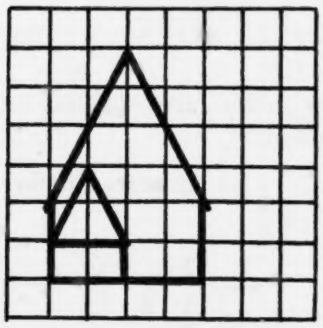


FIGURE 3

3. Logic Game

Four logic games are set out. Each set contains a large bristol board sheet with several shapes drawn on it, four different colours of yarn and a set of cards in the same four colours as the yarn. The cards may be classified according to colour (red, green, yellow or black), size (large or small) as well as shape. Students develop their own methods of placing the shapes and yarn on the bristol board (Fig. 4).

4. Geometry Lotto

This game is played much like Bingo except that colour and shape are used instead of numerals. As flash cards are held up by one student, other students cover the corresponding colour of the same shape with plastic discs.

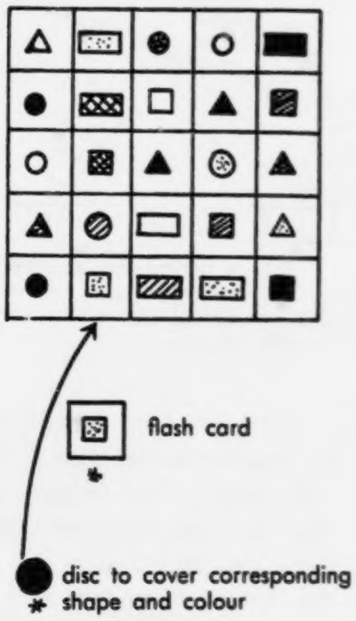

flash card
*
disc to cover corresponding
* shape and colour

FIGURE 5

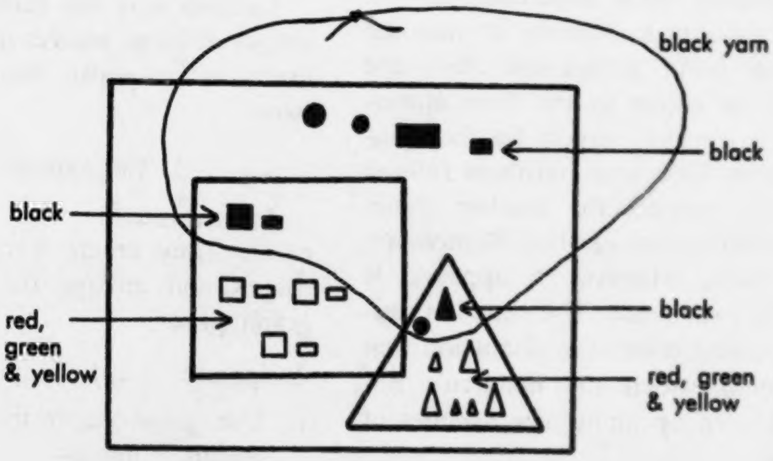
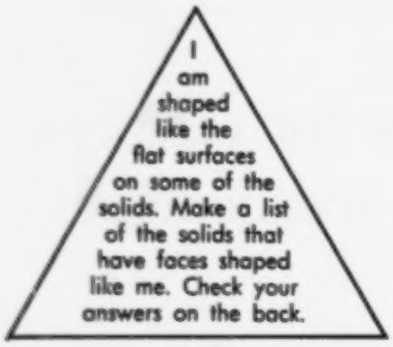

black yarn
black
black
black
red,
green
& yellow
red, green
& yellow

FIG. 4.—All other shapes remain outside the set.

5. FACE FINDING, VIA ASSIGNMENT CARDS

At this activity two people work from each assignment card. The cards are placed on the table with a set of solids.



I am shaped like the flat surfaces on some of the solids. Make a list of the solids that have faces shaped like me. Check your answers on the back.

FIGURE 6

6. IDENTIFYING SOLIDS BY FACE SHAPES

Students begin at point A and walk around the table from Card A to Card I listing the solids from which the surfaces (both flat and curved) have been traced. Then they repeat the rotation, this time checking the answers on the backs of the cards.

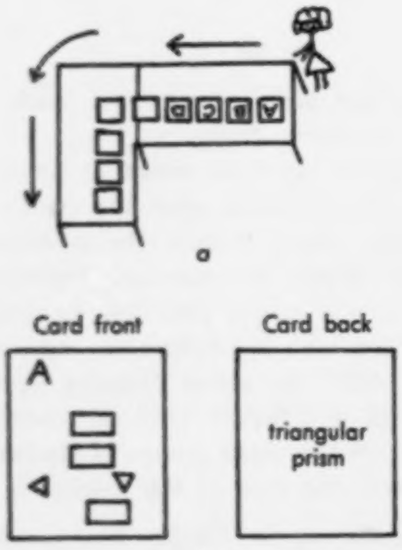


FIGURE 7

7. PERIMETER OF FACES

The pupils use yarn to find the distance around the faces of the solids, and fasten the yarn to a graph with gummed mosaic shapes.

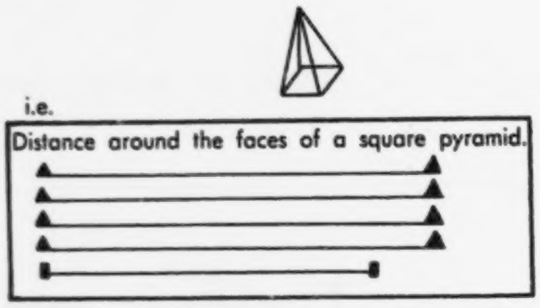
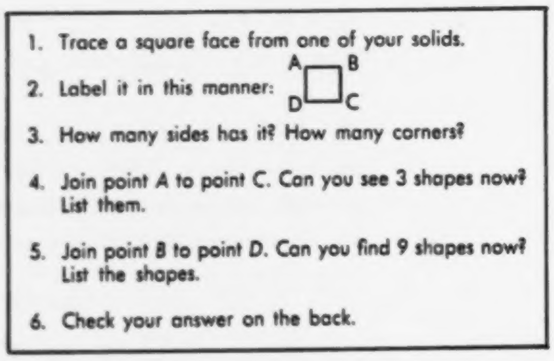


FIGURE 8

8. CREATING NEW SHAPES

Pupils work with solids, rulers and assignment cards similar to the following:



1. Trace a square face from one of your solids.
2. Label it in this manner: A B D C
3. How many sides has it? How many corners?
4. Join point A to point C. Can you see 3 shapes now? List them.
5. Join point B to point D. Can you find 9 shapes now? List the shapes.
6. Check your answer on the back.

FIGURE 9

Conclusion

In the 1966-67 session, the teachers trained in Toronto in July 1966 will carry the project further by experimenting with the teaching of the topics and concepts mentioned earlier. There will be three phases to the experiment, each for ten weeks' duration. In the fall, the material will be taught for ten weeks in western Ontario. It will then be revised, and tried out in modified form in eastern Ontario in the winter term, again for ten weeks. Finally, after a second revision, it will be taught for a further ten weeks in northern Ontario in the summer term.

By means of these successive revisions, it is hoped that the feasibility of both content and approach will be put strongly to the test. All those involved look forward to the experimentation with intense interest and a fair measure of confidence in the laboratory approach.

Forum on teacher preparation | *Francis J. Mueller*

Topics in geometry for teachers—a new experience in mathematics education

CAROL H. KIPPS
University of California, Los Angeles, California

Carol Kipps, besides teaching courses at the University of California, Los Angeles, is highly active with in-service projects, among them the Madison Project, the California Conference for Teachers of Mathematics, and UCLA Extension.

Can teachers capture by themselves the excited enthusiasm shown by children in classes sponsored by such curriculum groups as the Madison Project or Nuffield Project? Can a teacher reared on lecture-drill-homework classes feel and show the drama inherent in "I do and I understand" activities, in peer-group discussions, and in concepts such as the concrete-ikonic foundation of abstraction? A new course at UCLA is focusing on these dynamic factors so that teachers will know their value from their own personal experiences and feelings.

Geometry is the vehicle, and grades K–8 is the level. Geometry was chosen because prospective teachers have little background in geometry and very often fear having to teach it. More and more geometry is being introduced in the elementary grades. Teacher-training research and the recommendations of professional organizations show that geometry is more troublesome than arithmetic or algebra.[1]

Modern curricula aimed at optimal sequencing capitalize upon the child's early curiosity about shapes, the relations between shapes, and patterns. Informal exploratory geometry provides the necessary basis for later symbolization and abstraction. Also, an active learning approach requires a different kind of teacher behavior. When small groups of students are involved, the role of the teacher is more demanding—and far more rewarding. As the teacher moves from group to group listening to the dialogue, she must consider when to ask a question, when to be silent, and when to withdraw altogether.

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[1] See for example: *Goals for Mathematical Education of Elementary School Teachers: A Report of the Cambridge Conference on Teacher Training* (Boston: Houghton Mifflin Co., 1967). *Course Guides for the Training of Teachers of Elementary School Mathematics*, rev. ed. (Berkeley: Committee on the Undergraduate Program in Mathematical Association of America, 1968). Carol Kipps, "Elementary Teachers' Ability to Understand Concepts Used in New Mathematics Curricula," THE ARITHMETIC TEACHER 15 (April 1968): 367–71. Marilyn Suydam, "Research on Mathematics Education, Grades K–8, for 1968," THE ARITHMETIC TEACHER 15 (October 1968):531–44.

Many people tend to teach the way they have been taught. This can be a virtue as well as a hazard. In the experimental class taught at UCLA during the winter quarter of 1969 and then repeated in the summer, the class was conducted in the same way that corresponding classes ought to be taught in the elementary school. For not only can the process be modeled, but the teacher can evaluate it from personal experience, choosing appropriate learning activities and peer groupings with greater insight and precision.

Method

The course began with a discussion of the goals pupils should achieve by the end of the eighth grade. These behavioral objectives free the teachers from complete reliance on the basic text and focus on individualizing the learning opportunities to achieve at the 100-percent level. The following objectives were suggested as basic and minimal.

SHAPES IN GEOMETRY

1. The child will name flat or space figures when shown a physical model or a pictorial representation of the following: triangle, quadrilateral (square, rhombus, trapezoid, parallelogram, rectangle), circle, ellipse, cube, cone, rectangular solid, sphere, prism, pyramid.
2. The child can show where he would measure a flat or space figure to find the length of its diagonals and its altitude. Also, the child can draw a line on a pictorial representation to indicate what he would consider the altitude or a diagonal of the figure to be.
3. The child will state whether flat or space figures have point (turning) symmetry or line (folding) symmetry and define these ideas.
4. Given a physical model, a picture, a verbal description, or a description in set language, the child will state whether the figure is open or closed and whether it is convex.
5. The child will construct a physical model or sketch and will describe essential properties of triangles and tetrahedrons; squares, rectangles, cubes, and rectangular prisms; circles, cylinders, cones, and ellipses; polygons, regular plane, and space figures.
6. The child can name, define, and represent the foundation elements of geometry: points, lines, line segments, rays, planes, and angles.

RELATIONS BETWEEN SHAPES

1. The child will pick out congruent shapes and verify his decision by fitting. He will define congruent figures as those that can be made to fit together and use the notation $\cong$ for congruent.
2. The child will classify from a set those figures that are similar.
3. The child can identify examples, list examples, and sketch examples of the following relations between geometric shapes: covering (tessellate), separating, inside, outside, on, and topologically equivalent.
4. The child will identify, construct, sketch, or use set notation to describe the possible intersection sets for lines; lines and plane regions: lines and solid figures; two plane regions; and a plane solid figure.
5. The child will locate a given point on the number line. Given a pair of coordinates (x, y) that belong to the set of rational numbers, the child will locate the given point on a number plane.

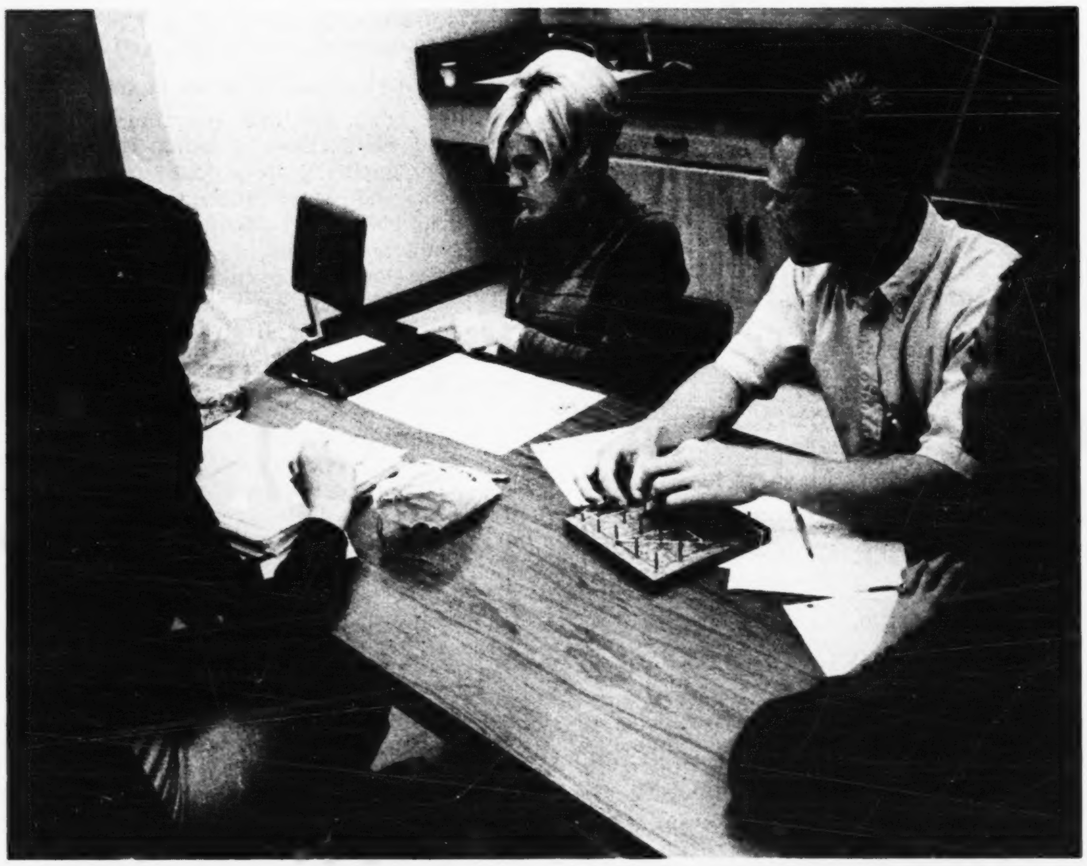
MEASUREMENT

1. Given practical problems involving measurement, the child will experiment (estimating, selecting appropriate units, gathering data from observations, constructing number scales, and computing) and will attempt to verify his solution in some fashion.
2. Given a standard unit, the child can roughly approximate and measure the length, area, and capacity of common objects in appropriate English and metric units. The allowable margin of error will depend on instruments used, if any. Since all measurements are approximate, the child can cite methods for reducing errors of measurement.
3. The child will verbalize that measurement is the process of selecting an appropriate standard unit and finding a number that compares the two. The child will state generalizations that follow, based on measurements and the tables or graphs in which the measurements are recorded.

LOGIC AND PATTERNS

1. Given one, two, or three criteria for selection, for example, color (e.g., red), shape (e.g., square), and size (e.g., large), the child can classify elements of a set in a way that shows which elements have none, one, two, or three attributes.
2. The child can determine the pattern of a sequence of figures or numbers, continue the sequence for at least three more elements, and verbalize the criteria that he is using.

The big change in method in these experimental classes was that the university students were asked to work together in small groups on learning opportunities designed for specific behavioral objectives. What lecturing there was had to do mostly



with learning theory—Piaget, Bruner, Gagné. While solving the problems, the students were encouraged to use many kinds of resources, e.g., concrete embodiments of various mathematics ideas such as Dienes Multibase Arithmetic Blocks, Cuisenaire rods, and geoboards. A curriculum library was available that included dictionaries, textbooks, and teacher guides from the experimental projects such as Nuffield and Madison Projects as well as those for standard texts. The instructor acted as a consultant, answering questions with questions and suggesting references. As discussions waxed about the definitions of words such as *diagonal* and *altitude,* their adequacy for the present two-dimensional problems or similar ones in three dimensions were debated. Preservice teachers enjoyed "playing teacher" with each other and using clues to draw out those more naive mathematically. In an in-service situation, the instructor would have many "assistants."

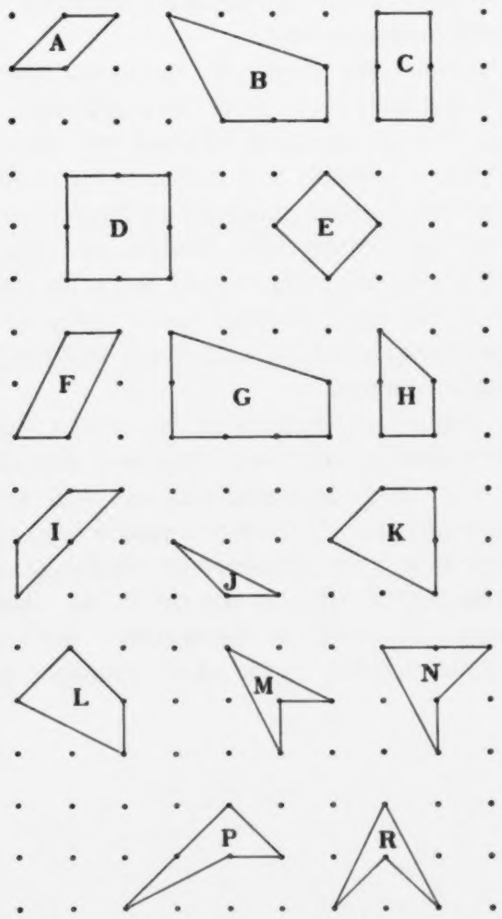
A second key difference from the usual university course in education was the use of evaluation as one of the learning activities. As part of the course, the students not only answered questions, but proposed them! Many educational taxonomies suggest that asking an insightful question aimed at a particular learning behavior is a much higher cognitive skill than answering questions. After solving a mathematics problem in their peer-group situation, students were asked to devise a suitable test item. At this point each person was asked to make an individual contribution, but by all means to consult with the group about it. While the test item was to be based on the stated objective, it might well include prior skills or knowledge and need not be a paper-and-pencil type of test item.

To illustrate this classroom activity, the following is a page taken from the class notes. The students were required to make their own geoboards and to bring them to class for this type of activity. First, as a teacher, each was to read the specific behavior objective. Next, as a pupil, each had to solve the problem, discussing it with members of his small group. Last, again in the role of a teacher, each participant was to read the sample test item and write another of his own based on the objective.

SAMPLE CLASS WORKSHEET

Specific behavioral objective.—Identify the diagonals of various plane figures and define the idea of a diagonal.

Learning opportunity.—Make the following shapes on your geoboard. Transfer each figure to dot paper and draw in all the diagonal lines using a red pencil. In which shapes are the diagonals of equal length?



Test item.—Define what you mean by the term diagonal. Draw a shape that has no diagonal and tell why it doesn't by applying your definition.

Test item.—Does a diagonal necessarily bisect the angle at that vertex?

Comments.—A follow-up discussion might develop the idea of whether the definition given would work for space figures, or for a line joining two vertices which is not a side.

Evaluation

This methods course in mathematics education at UCLA is organized primarily to develop—

(1) skill in planning and evaluating learning opportunities in mathematics for pupils;

(2) skill in using the Socratic approach;

(3) knowledge and skill in applying basic concepts of informal geometry.

At frequent intervals, test items written by the students on the worksheets were reviewed by the instructor and returned with suggestions or comments. Grades were not given for these worksheets, because it was hoped that the content and experiences would foster interest in the learning activity rather than in some extrinsic reward. Relevance to the specific objective and mathematical correctness were checked. Creative style and elegance were noted with positive comments. It was a matter of considerable delight and astonishment to the instructor when not one member of the class of 44 asked about getting a "grade." This attitude is most appropriate to a study of teaching mathematics, for the "new math" was introduced not merely on the claim that it represented more important content but equally on the argument that it would build a new spirit of inquiry and creativity.

Grades for the course were assigned on the basis of an examination focusing on the methodology. Here is a sample question:

> Select one of the objectives from above and describe three learning experiences that you could provide to enable children to achieve the objective. Write one learning experience for each of the following levels:
>
> Enactive (sensory-motor)
> Ikonic (representational)
> Symbolic (abstract)

Continual feedback of geometric content was possible during the class periods, since the instructor could spend time with a small group or an individual student at no expense to the rest of the class. It is the rare student who will display his ignorance in a conventional classroom; but in the small-group approach, important questions are readily raised and discussed. When a student works in a small group, he finds it much easier to express his confusions enough to enable others to help him.

A term paper on a mathematics topic selected by the student was required by the course. The basic text was broadly representative of arithmetic, algebra, and geometry. Prior to the use of the geometry notes and discussion "in fours," few students had selected a topic from geometry for this paper. It is of note, then, that 28 out of the 44 felt comfortable with geometry and did their paper in this area.

Conclusion

Prospective teachers have little background in geometry and very often fear having to teach it. It is apparent that many potential teachers exposed to this approach will come away with a good feeling about geometry, some of the confidence needed to teach geometry, some exploratory activities they can use with children, and a much broader knowledge than that normally obtained from a straight geometry or mathematics course.

Some very important questions cannot yet be answered. Does this approach to teacher training tend to make the content easier to retrieve and to reconstruct? Will teachers include geometry in their lessons and use appropriate manipulative apparatus and resources beyond the pupil text? A follow-up is planned in the form of a questionnaire at the end of the first teaching assignment.

Student evaluations of the format were enthusiastic, and they expressed attitudes toward studying mathematics as well. One student wrote, "I think you should keep the course as it is, because the workshop atmosphere is what is needed in the classroom, and we as prospective teachers should practice in the same atmosphere."

2

Instruction—Techniques

Instruction! This is the key word in any teacher-learner relationship. The involvement of the learner with the subject matter is directly related to the teacher's instructional procedure. For genuine involvement, good instructional techniques and principles must be generated.

In this book the articles on instruction are divided into two groups according to whether they deal with *techniques* or *rationale*. These are arbitrary categories, designed to facilitate reader interpretation. Techniques for instruction are the vital elements that lead to and are derived from involvement. A rationale for techniques is often difficult to express. Some fine teaching may occur fortuitously, but it is generally true that good teaching grows out of good rationale and commendable planning. Without a rationale, a technique is scarcely better than a gimmick, time-occupier, or dramatic presentation.

The involvement of the learner is a function of instruction. Without effective instruction, few children discover the joy and excitement of mathematics. The techniques and methods of teaching geometry lend themselves to a creative, mathematical involvement of the learner.

Early instruction in geometry should be termed *environmental geometry* because much of the learning is fostered by challenging the child to discover his surroundings. Born into and enclosed in a three-dimensional world, the child has ready access to geometric models. Under the skillful hands of a good teacher, he can be moved to the excitement of informal geometry and later led to the thrill of elegant geometric abstraction.

Mirrors, models, toys, and Möbius bands are some of the tools for environmental geometry. The following articles present accounts of teacher-proven techniques that use these and other readily available materials. Reporting the use of a mirror and some cleverly developed cards, Walter describes an approach to symmetry that can be used with older students as well as very young children. Spatial perceptions and ideas of geometric transformation are by-products of this approach.

Alspaugh pairs two mirrors to give a kaleidoscopic effect when lines and models are viewed in them. Interesting ideas of symmetry, geometric form, and pattern can be developed using this approach. The content discussion of symmetry by Dennis nicely balances the techniques and methods of Walter and Alspaugh. The teacher will need to study the ideas of symmetry to be effective in teaching them. Interestingly enough, Forseth and Adams reverse the traditional method of using art to teach geometry. They first introduce various transformations and then use them as a means to develop art form and pattern. The fusing of art and geometry is clearly presented.

Topological ideas are somewhat new to the study of geometry. Piaget has shown that children develop intuitive ideas of topological geometry before those of traditional Euclidean space. D'Augustine illustrates how a teacher may develop topological generalizations using common experiences of the child. His development is followed by that of Clancy in which the Möbius band is used as a means to discuss dimension. From the study of dimensions, it is but a small step to the one- and two-dimensional creatures of fanciful writing developed by Carrol's class. Carrol relates the imaginative expression of her children to the ideas of dimension and geometric abstraction.

Milk cartons are reasonably easy to obtain. With these convenient materials, a teacher guided by Walter's second article can develop excellent lessons dealing with nets, patterns, and generalizations. As the reader inspects these ideas, questions of volume and volumetric relationship may be raised. Vincent, in a very brief note, gives a fine method for comparing the volumes of the cylinder and the cone. Her technique is easily modified for other volumetric as well as area comparisons.

Any survey of instruction in geometry would be incomplete without some mention of the useful geoboard. Smith, in one of the earliest articles on techniques, discusses and illustrates the geoboard (pegboard). Liedtke and Kieren also discuss the use of the geoboard in the context of early childhood education. The results obtained suggest experiences that could be used for preschool and kindergarten education. Another device that can be used for developing ideas of shape, congruence, vocabulary, and relationships is a Tinkertoy set. Richards shows how Tinkertoys can be effectively used in geometry. As with other devices, the geoboard and Tinkertoy set can be best used by the teacher who knows the content of the geometry to be taught. Jackson provides some of this content in his discussion of congruence and the relationship of geometry to measurement. In the final selection in this grouping of articles, Jackson weds the technique to the subject-matter content.

Taken together, these readings indicate the direction a teacher may go. The ideas expressed by the authors are creative and provide motivation for the development of a teacher's own approaches. Those who deal directly with children will find this selection of articles of particular importance and practical usefulness.

An example of informal geometry: Mirror Cards*

MARION WALTER
Educational Services Incorporated, Watertown, Massachusetts

Marion Walter is a part-time mathematics instructor at the Harvard University Graduate School of Education. She is on the staffs of Educational Services Incorporated in the Elementary Science Study and the Cambridge Conference on School Mathematics. She teaches mathematics to the students in elementary school education at the Harvard Graduate School of Education.

The need for informal geometry, especially in the earlier grades, is being recognized by educators, psychologists, and mathematicians. The Mirror Cards were created by the author to provide a means of obtaining, on an informal level, some geometric experience that combines the possibility of genuine spatial insight with a strong element of play.

The basic problem posed by the Mirror Cards is one of matching, by means of a mirror,[1] a pattern on one card with a pattern shown on another card. For example, can one, by using a mirror on the card shown in Figure 1, see the pattern shown on the card in Figure 1a?

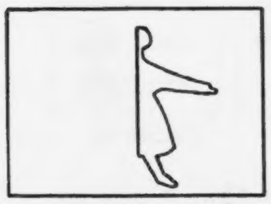


FIGURE 1

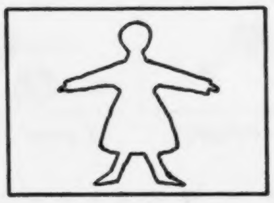


FIGURE 1a

The problems range from the simplest, such as the one shown above, to more difficult ones, such as the one shown in Figures 2 and 2a.[2] Some patterns are possible to match and others are not.[3]

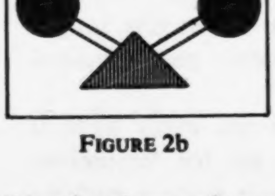


FIGURE 2

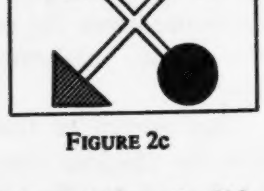


FIGURE 2a

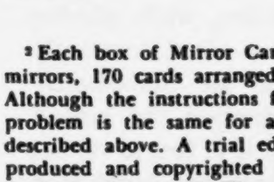


FIGURE 2b

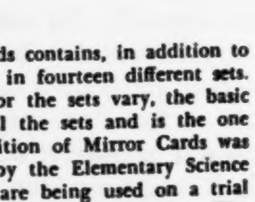


FIGURE 2c

Using the mirror on the card shown in Figure 2, which of the patterns shown in Figures 2a, 2b, and 2c can you make?

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* This work was begun while the author was working during the summer of 1963 with the Elementary Science Study, a project supported by grants from the National Science Foundation and administered by Educational Services Incorporated, a nonprofit organization engaged in educational research. She would like to thank the members of the group she worked with that summer and the group in optics of the previous summer for their help and encouragement; she is especially grateful to Professor Philip Morrison, Mrs. Phylis Singer, and Mrs. Lore Rasmussen.

[1] The reader should have a small rectangular pocket mirror handy before reading on.

[2] Each box of Mirror Cards contains, in addition to mirrors, 170 cards arranged in fourteen different sets. Although the instructions for the sets vary, the basic problem is the same for all the sets and is the one described above. A trial edition of Mirror Cards was produced and copyrighted by the Elementary Science Study in June 1965. They are being used on a trial basis in over 250 classrooms around the country. The author would like to acknowledge the help received from Mrs. A. Naiman, Mrs. F. Ployer, and Mrs. J. Williams in editing the guide and producing the cards.

[3] The position of the pattern relative to the edge of the card is to be ignored.

We have noticed that the children usually find the colors and shapes pleasing and enjoy the challenge presented by the cards. They do not think of this work as "mathematics," and they often find the cards stimulating over and above the actual geometry involved. The cards may be a means of reclaiming the children who already dislike mathematics or are bored or frightened by it. The cards do not call for verbal response from the children, and no mathematical notation is needed. Closer connection with science and mathematics classes will be explored by the author in the future, since the cards can give insight into some mathematical and physical principles.

One advantage that the cards have is that the children can see for themselves whether or not they have made a pattern. They don't need to resort to authority to check whether they have solved the problem correctly. In addition, while playing with the cards they are, in effect, constantly making predictions and are immediately able to test these predictions and amend them, if necessary; and it is fun to do so! Thus, while working with the cards they should gain confidence in their own powers and learn through experience the nature of the scientific method.

While moving the mirror around on the cards, the children notice and experiment with the position of object and image in relation to the edge of the mirror. The player can decide where to place the mirror; and he soon learns that he can control its position, but that for any given position of the mirror he cannot control the position of the image!

The students also learn that a mirror does not carry out a translation. (See Figs. 3, 3a, 3b.)

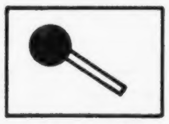


FIGURE 3

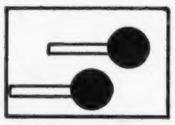


FIGURE 3a

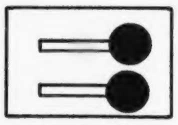


FIGURE 3b

Can one by using a mirror on Figure 3 make the patterns shown in Figures 3a and 3b? — alas, the mirror does not carry out a translation!

They learn by experience that congruence of two parts is a necessary but not a sufficient condition for a pattern to be made by use of a mirror. Most children do not know the expression "symmetric with respect to a line" or "reflection in a line." They may, nevertheless, by using the cards, gain experience that will enable them to understand the concepts that these expressions describe. This does not imply that they could give, or should be expected to give, a formal or verbal definition of these expressions. Eventually they do notice that for a pattern to be reproducible by use of a mirror, it must have two parts that lie on either side of some line and that these must "match exactly." They soon learn, for example, that the pattern shown in Figure 4a cannot be made from the pattern in Figure 4, and they probably have a good feeling for why this is so.

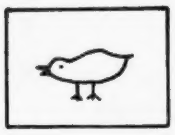


FIGURE 4

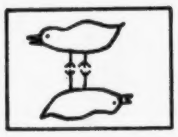


FIGURE 4a

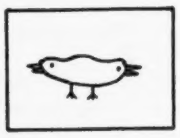


FIGURE 4b

Pattern 4a cannot be obtained from 4. What about the pattern in 4b?

The cards provide opportunity to practice recognizing congruent figures and selecting parts of figures congruent to another.

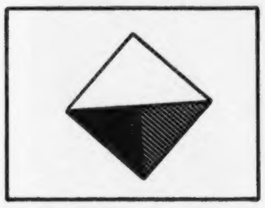


FIGURE 5

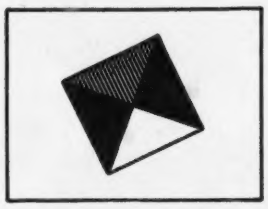


FIGURE 5a

Where must you place the mirror in Figure 5 to see the pattern shown in Figure 5a?

The children must be observant, not only about a shape and the position of that shape, but also about its colors. Some of the patterns match in shape but not in color.

They may also notice a variety of geometric properties of figures. Consider, for example, the circle. By putting the mirror on a diameter they can see the whole circle. More than that, any diameter will do and any chord not a diameter will not do. This may give young children their first feeling for a diameter of a circle, long before they know the word "diameter."

With the diamonds (see Fig. 6) they notice that there are two places where the mirror may be placed to enable them to see the whole diamond. On the other hand, the pattern shown in Figure 7 does not have this property—to the surprise of many!

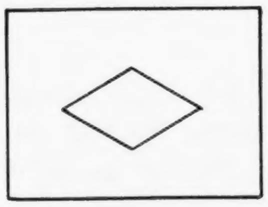


FIGURE 6

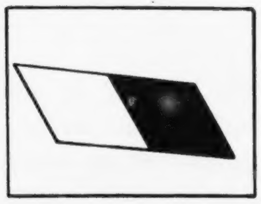


FIGURE 7

Or, again, take the triangle (see Fig. 8): the children may notice that the effect of putting the mirror along *AB* is in some way "the same" as that of putting it along *BC*, but that it is quite different from that of putting it along *AC*. What about *BD*?

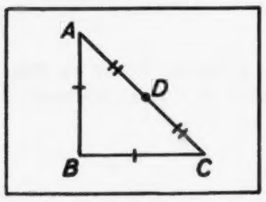


FIGURE 8

Other patterns on the cards, such as the ladybugs, arrows, etc., can be explored in similar ways.

For a few cards the children can obtain patterns that look somewhat like the one required but are not congruent nor actually similar in the mathematical sense. I intend to devise cards where congruent and similar patterns are obtainable, and similar but not congruent ones.

Unfortunately, none of the present cards have circles with arrows on them to show, perhaps more clearly, that the orientation gets reversed under a mirror mapping or reflection. Thus Figure 9 becomes Figure 9a.

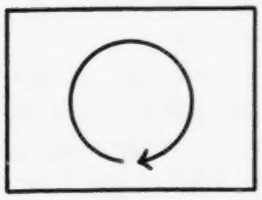


FIGURE 9

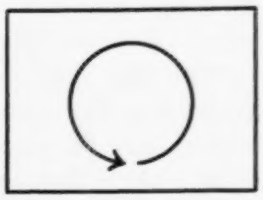


FIGURE 9a

The concept of orientation is, of course, brought out by the cards, although the arrows are not used for this purpose. Often patterns with orientation reversed and not reversed are included to make the idea more obvious. Examples taken from the ladybug and the circle set are shown below.

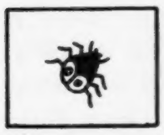


FIGURE 10

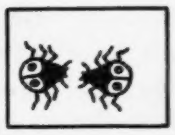


FIGURE 10a

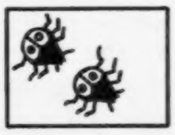


FIGURE 10b

Can one by using the mirror on Figure 10 obtain the patterns shown in Figures 10a and 10b respectively?

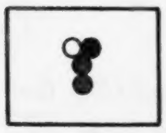


FIGURE 11

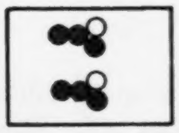


FIGURE 11a

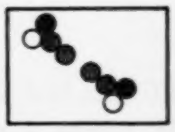


FIGURE 11b

Can one by using the mirror on Figure 11 obtain the patterns shown in Figures 11a and 11b respectively?

The fact that a mirror does not carry out a rotation in the plane is often masked by the symmetry of the figure. For example, one can make Figure 12a from Figure 12, but not Figure 13a from Figure 13.

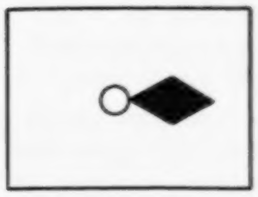


FIGURE 12

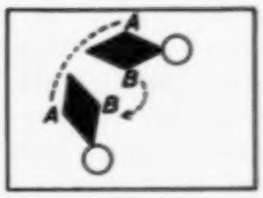


FIGURE 12a

The imagined placement of points "A" and "B" illustrates the fact that the mirror does not "rotate" the figure. Actually the mirror "flips" the image. (Points "A" and "B" are not marked on the actual cards.)

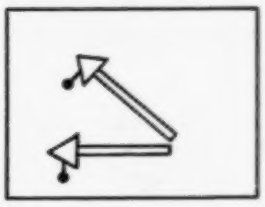


FIGURE 13

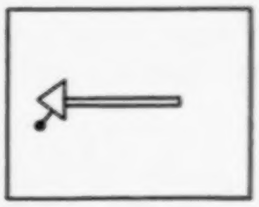
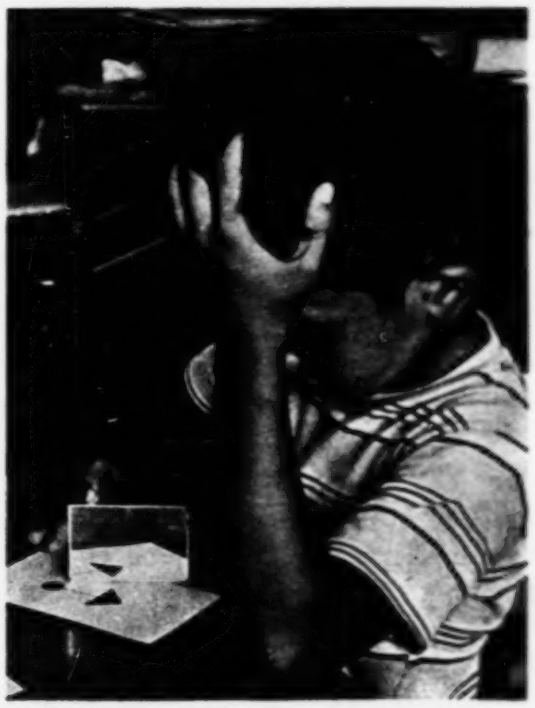


FIGURE 13a

The cards may be used at any age level. They have been used by children as young as five and by sophisticated professional scientists or mathematicians. It is interesting to note that some adults who "know



all the rules" verbally (such as "There must be a line of symmetry" or "Image distance = object distance") often have more difficulty in working through the sets than children who have not yet memorized such phrases. The one barrier to the effective use of the cards by adults appears to be an ingrained habit of respect for authority. Adults often do not want to rely on their own ability to *see* whether they have made a pattern correctly.

When the children find the problems becoming too easy, they may want to add the rule, "You may put the mirror down only once for each pattern," so that all the trial and error must go on in their heads. They may wish to make some of their own cards. When, as happens often, children are able to predict without using a mirror at all whether a pattern can or cannot be made, they will have a clear demonstration of the power of reasoning based on experience—i.e., that it is possible to predict results with confidence by thinking rather than doing! (And they are able to check their thinking if they wish.) In this way they are savoring an essential part of the nature of rational thought.

There are many questions that still need to be answered. I mention just a few. Will use of the cards make children more observant about other geometric patterns? Will it enable them to see figures within figures more easily? Does it improve their ability to visualize? Will they be able to describe patterns more clearly? Will it help or hinder children with reading difficulties?

EDITORS' NOTE. *Current information about Mirror Cards (#18418) and Mirror Cards Teachers' Guide (#18417) can be obtained from Webster Division, McGraw-Hill Book Co., New York.*

Kaleidoscopic geometry

CAROL ANN ALSPAUGH

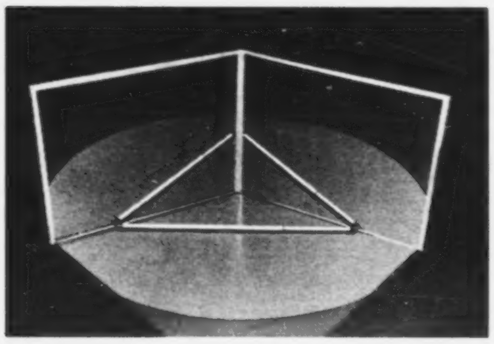
Carol Ann Alspaugh is doing research for her doctoral dissertation at the University of Missouri, Columbia, Missouri. She formerly taught mathematics and computer programming at Southwest Missouri State College and at other colleges and universities.

Kaleidoscopic geometry is an interesting type of mirror geometry that could be utilized to introduce geometrical topics such as regular polygons, coordinates of points in a plane, reflections, and symmetry. Most children enjoy the possible explorations offered by this geometry, which would make it useful to the teacher desiring to develop interest and motivation when introducing new materials.

The physical materials needed for this geometry are simple and inexpensive. Two mirrors are hinged together either by gluing them to one piece of cardboard or by using masking tape. In this way, the mirrors may be set up at any desired angle on a table or folded together to facilitate easy storage. Unbreakable mirrors are now available on the market and could be used.

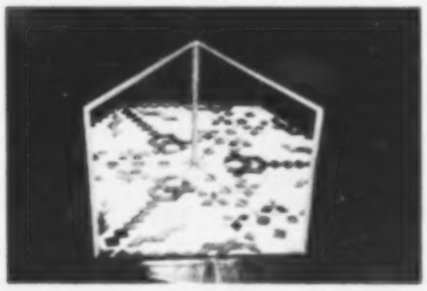
Students at the fifth- or sixth-grade level would be ready for this geometrical experience. By this time, the student would have studied various polygons, be familiar with the use of a compass and protractor, and know that there are 360° in a circle. An interesting point about this unusual geometry is the seemingly unlimited number of mathematical concepts that can be illustrated as the angle between the two mirrors is allowed to change. When these mirrors are placed in front of them, most students and teachers will find some fascinating challenges as they look into the new world appearing before their eyes. A few of the concepts that are possible to illustrate are presented on the next page. However, for maximum learning and appreciation on the part of the student, he should be allowed to discover these concepts himself by physical manipulation of the materials in a laboratory situation.

1. Using a wood cube (or a similar object) placed in the center of the table between the positioned mirrors, it will be observed that as the angle θ between the mirrors increases, the number of cubes (images) decreases to a minimum of 2 when $\theta = 180°$, and increases without bound as $\theta \to 0°$.
2. The number of images, I (where I includes the original object), is related to the angle θ by the following formula: $I \cdot \theta = 360°$.
3. When $\theta = 120°$, three images are seen, and the following geometrical constructions are possible by drawing *one* line segment on the table between the mirrors (a ruler or extendable curtain rod could be used as a line segment):



a. equilateral triangles (any other kinds of triangles are impossible to construct)
b. circles (by making one arc between the mirrors)

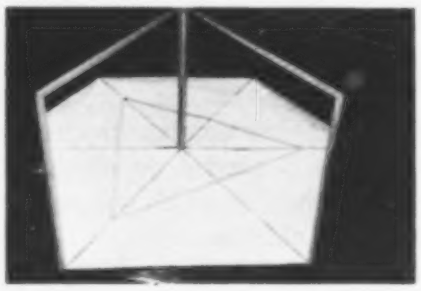
It is interesting to discover that squares, pentagons, etc., cannot be constructed when the angle between the mirrors is 120°. When the angle is varied, it will be noted that the constructions that can be made will also vary. However, it is always possible to construct a circle, and any polygons that can be constructed will have equal sides.



4.	θ	I = images	Possible Constructions
	180°	2	parallel lines, circles
	120°	3	triangles, circles
	90°	4	squares, parallelograms, parallel lines, circles
	72°	5	pentagons, circles
	60°	6	hexagons, triangles, circles
	51 3/7°	7	septagons, circles
	45°	8	octagons, squares, circles
	40°	9	nonagons, circles
	36°	10	decagons, pentagons, circles

The preceding table might be an example of a student's summary of the constructions he discovered that were possible to make. Of course, the table could be extended.

5. With the mirrors at an angle of 90°, graphing in the coordinate plane can be nicely illustrated. For this purpose, the teacher or student should cut a piece of poster paper to lie on the table and fit between the mirrors. Then a coordinate system may be ruled off on the paper, using the point where the mirrors meet as the origin. As a point is located on the paper, the reflections will simultaneously locate the points in the other three quadrants.



Many types of poster paper overlays could be designed by the creative student and teacher to illustrate mathematical and artistic ideas. Children who would be too young to make some of the suggested tables and generalizations would enjoy creating or copying kaleidoscopic designs and polygons using colored parquetry blocks, straws, sequins, etc.

It is hoped that this paper has suggested that the principles underlying the design of a child's toy kaleidoscope have many teaching possibilities for elementary mathematics.

Informal geometry through symmetry

J. RICHARD DENNIS
University of Illinois

J. Richard Dennis is coordinator of the Hawaii-UICSM College-School Cooperative Project in Honolulu, Hawaii.

The geometry component of the elementary school program is the basis of much discussion today. Many of the efforts in this area of mathematics have approached elementary school geometry from the point of view of Euclid's postulates. Another approach to geometric topics is based on the idea of symmetry.* A unit using this approach would involve five to six weeks of class time and would do much to augment the elementary school geometry program.

The first idea that we need to make clear is the meaning of the phrase "exactly alike" as it is used in geometry. Students already have opinions on the meaning of this phrase, but their opinions frequently differ. Agreement is needed on an experimental test, the results of which will be acceptable in cases of differing opinions.

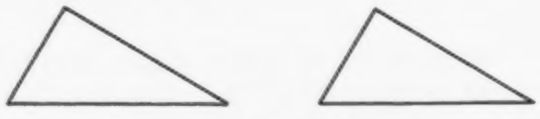


FIGURE 1

A pair of figures that appear to be exactly alike are shown in figure 1. How can we tell for sure? The test that we agree upon is to trace one figure and then try to match the tracing with the other figure. This match need not be achieved in any particular position or orientation. It may be possible to achieve a matching in several positions. It must be possible to achieve a matching in at least one position (fig. 2).

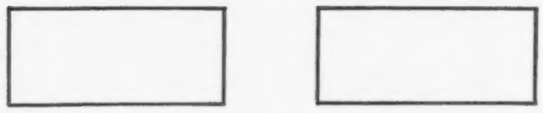


FIGURE 2

When it is possible to exactly match a tracing of one figure with another figure we say that the figures are *congruent*, and the various matchings are called *congruences*.

Teachers should create many opportunities for children to experiment with the "trace and try to match" process described above. It is from such experiments that basic intuitions about congruence are derived. These intuitions will become a foundation for the discovery and exploration of more complicated properties of geometric figures. For example, by experimenting with a tracing, congruences can be immediately separated into two types:

1. *face-down* congruences for which the tracing must be turned over to make it match (fig. 3)

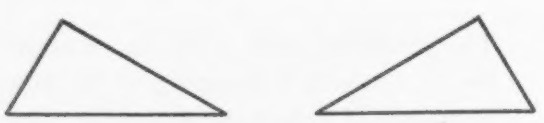


FIGURE 3

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* The ideas presented here grew out of my association with the University of Illinois Committee on School Mathematics. I wish to express my gratitude to Professor Max Beberman for the opportunity to participate in the UICSM activities and for his generous counsel.

2. *face-up* congruences for which the tracing is not turned over, just moved around to make it match (fig. 4)

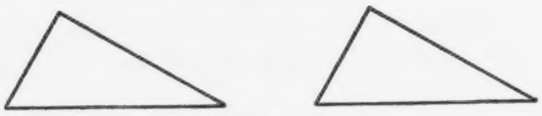


FIGURE 4

Having distinguished between these two types of congruences, we concentrate on the effects of each of them. We ask questions like the following:

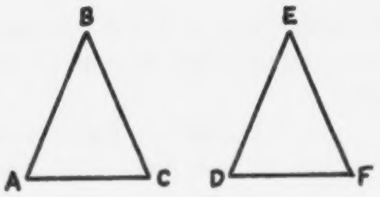


FIGURE 5

1. For the face-up congruence of these figures, with what part of triangle *DEF* does the tracing of segment *AB* match?
2. For the face-down matching, with what part of triangle *DEF* does the tracing of angle *ACB* match?
3. Again for the face-down matching, with what part of triangle *DEF* does the tracing of segment *AC* match?

Some important facts to observe are these:

1. For both matchings, the tracing of point *B* matches point *E*.
2. For both matchings, the tracing of segment *AC* matches segment *DF* but the tracings of individual points of the segments do not match in the same way.

Work of this nature gives an introduction to the notion of corresponding parts for a congruence, and, of course, since the tracing is used to match these parts, corresponding parts of congruent figures are congruent.

The next step in this development is to apply the notions of congruence and corresponding parts to a single figure rather than to two figures. Specifically we study *self-congruences* of a figure. Again these are of two types—face up and face down.

Figures with face-down self-congruences have a very important property. There is a line each of whose points corresponds with itself for that face-down matching. For example, consider the triangle in figure 6.

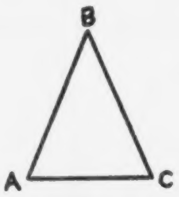


FIGURE 6

For the face-down self-congruence of this triangle, point *B* corresponds with itself. There is also a point of segment *AC* that corresponds with itself for this matching. In fact, if we draw a line through these two points, each point of that line corresponds with itself for the face-down congruence. It is such lines that we shall call ***lines of symmetry***.

Through this definition, each line of symmetry is associated with a face-down congruence of a figure. So when given an exercise such as to find all lines of symmetry for a particular figure, the student need only count the face-down matchings of a tracing. After a little practice, students easily move to the stage of just thinking about the tracing. It is important to note, however, that when all else fails, a *tracing* will make answers to questions quite obvious.

We are now ready to begin our study of triangles. It is assumed that many of the figures used in the previous work have been triangles. Students should see and experiment with triangles with no lines of symmetry, triangles with one line of symmetry, and triangles with three lines of symmetry. An important exercise is to have students try to sketch a triangle with exactly two lines of symmetry. Another important exercise is to try to sketch a triangle with a symmetry line that does not go through a vertex.

You will recognize the triangles with one (or more) lines of symmetry as those usually called *isosceles* triangles. Those with three lines of symmetry are usually called *equilateral* triangles. Having looked at the possible line symmetries for triangles, students are in a position to find properties of each type of triangle. For example, consider a triangle with one line of symmetry, i.e., one face-down self-congruence (fig. 7).

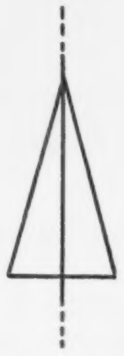


FIGURE 7

1. There is a pair of congruent sides, because for the face-down self-congruence the tracing of one side of the triangle matches another side of the triangle.
2. There is a pair of congruent angles, because for the face-down self-congruence the trac- of one angle matches another angle of the triangle.
3. The symmetry line goes through the middle point of one side, because for the face-down self-congruence the tracing of one part of this side matches the other part of this side.
4. The symmetry line bisects one angle (for similar reasons).
5. The symmetry line "divides" the triangle into two congruent regions (for similar reasons).

At this stage some other important questions should be considered:

1. Could a triangle have a pair of congruent sides without having a line of symmetry?
2. Could a triangle have a pair of congruent angles without having a line of symmetry?

For each of these questions, evidence is easily gathered from an experimental sketch and a piece of tracing paper. The properties of triangles with three lines of symmetry are presented in a like manner.

Before classifying quadrilaterals it is convenient to introduce the notions of perpendicular and parallel lines. For perpendicular lines we look at pairs of lines and ask: In what cases is one line a line of symmetry for the other? For example, here are a dashed line and a solid line (fig. 8).

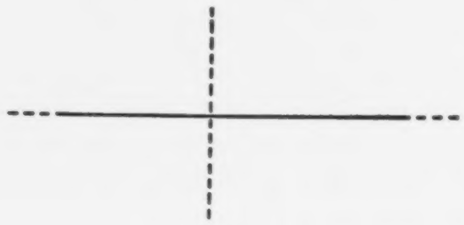


FIGURE 8

The dashed line is a line of symmetry for the solid line. We can also show this with a tracing.

Here it is important to have made clear the idea that, at best, pictures of lines leave much to be desired. For figures like triangles, lines of symmetry appear to cut the picture in half. For lines, it is no longer possible to judge lines of symmetry by checking to see if the picture is cut in half. An important observation is that when one line is a line of symmetry for another, the two lines make square corners with each other. We say that two lines are *perpendicular* whenever one is a line of symmetry for the other.

For parallel lines we again look at pairs of lines, but this time we ask if the lines have a line of symmetry in common. Again we can use a tracing (fig. 9). Parallel lines are those lines which do have a line of symmetry in common.

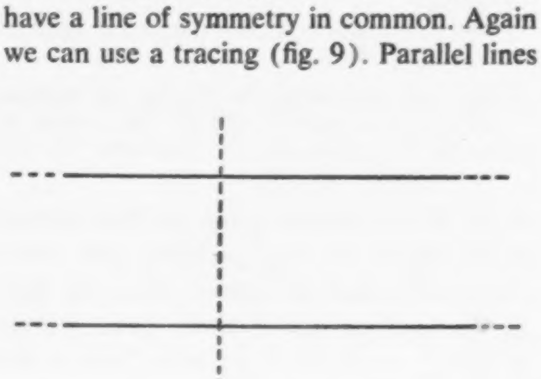


FIGURE 9

The face-down self-congruences gave us lines of symmetry. One type of face-up self-congruence is particularly important for the study of quadrilaterals. Sometimes a figure has a face-up self-congruence for a half-turn of a tracing. When this happens there is a point which corresponds with itself. Such a point is called a *center* or *point of symmetry* (fig. 10). Again it is important to examine several figures for points of symmetry, and to look for corresponding parts under these half-turn self-congruences.

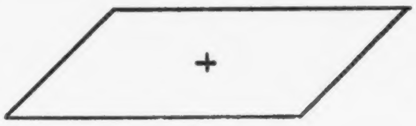


FIGURE 10

As we begin the study of quadrilaterals, we notice one important feature not found among triangles. Quadrilaterals may have lines of symmetry that do go through vertices or lines of symmetry that do not go through vertices (fig. 11). So we introduce the phrase *diagonal symmetry line* for those that do go through vertices, and the phrase *nondiagonal symmetry line* for those that do not go through vertices.

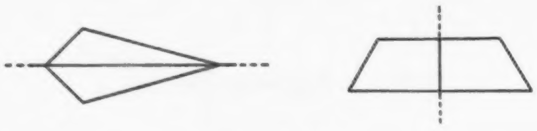
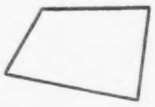


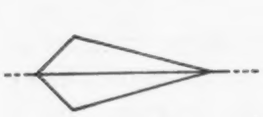
FIGURE 11

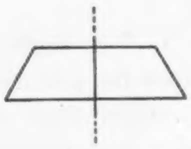
When we classify quadrilaterals, we find those with:

1. No lines of symmetry.

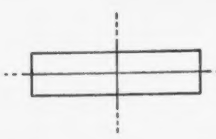


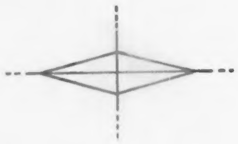
2. One line of symmetry.



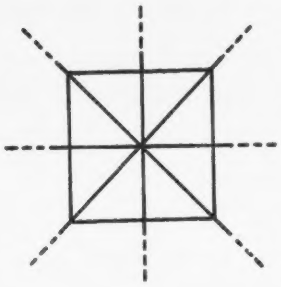


3. Two lines of symmetry.

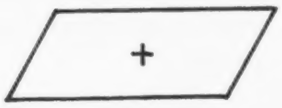


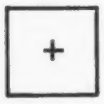


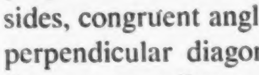
4. Four lines of symmetry.

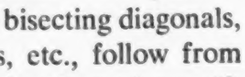


5. A point of symmetry.









Notice that when quadrilaterals have two or more lines of symmetry they also have a point of symmetry. There are no quadrilaterals with exactly three lines of symmetry.

As was the case for triangles, the usual properties about congruent and parallel sides, congruent angles, bisecting diagonals, perpendicular diagonals, etc., follow from the corresponding parts for the various self-congruences.

This has been a brief discussion of the topics involved, an appropriate sequence for these topics, and some sample questions at the various points of the development. Probably the most important word of caution to the teacher is to allow plenty of time for making experimental sketches and for conducting tracing experiments. Eventually many students reach a point where they can answer questions by merely conducting a thought experiment, but very few of them begin at this level.

Symmetry

SONIA D. FORSETH and PATRICIA A. ADAMS
University of Minnesota, Minneapolis, Minnesota

Sonia Forseth is the art director at the Minnesota Mathematics and Science Teaching Project (MINNEMAST).

Patricia Adams is an editor for the MINNEMAST Project and is working on her master's degree in journalism.

Combining art projects with math and science is an ideal way of promoting creativity in children. They become aware of how to use math and science; they learn to make and solve creative problems; and they have fun while doing it!

Symmetry is a basic geometric concept and a common, pleasing design found in nature, especially in leaves and flowers. Art activities using concepts of symmetry can be used with almost any grade and with every child, from the slowest to the brightest. The children will have fun making fascinating designs, while at the same time learning craftsmanship skills and achieving ideas for various types of symmetry. With some modifications, these activities can be used with any given grade.

Begin by showing the children examples of pictures of strip patterns in such things as architecture, pottery, tapestries, fabric, leaves, flowers, caterpillars, or centipedes. Older children may enjoy studying symmetry in physics, chemistry, and biology. Molecular structure and X-ray crystallography provide examples of geometric symmetry.

For materials, the children will need sheets of paper (such as construction, butcher, or bond), 3-by-5 inch index cards, color crayons or felt-tip pens, tempera paint, scissors, and common pins.

Here are seven basic strip patterns. The process of making each pattern is called an "operation" because you follow certain rules. The patterns vary in difficulty, so choose whichever one best suits your class.

1. *Repeating Patterns*, a very easy one, is made by simply moving the stencil a fixed distance each time you trace it (fig. 1). Have each child make a stencil by cutting an asymmetrical design in an index card. The children can save the cut out part for other patterns.

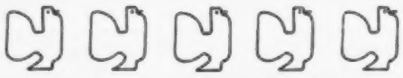


FIG. 1. Repeating Patterns

2. *Translation Reflections* is shown in figure 2. Trace the design, flip the stencil forward, and trace it again so that it looks like a reflection of the first design. If you are teaching this pattern to young children, have them draw a line across their papers (fig. 2). Ask them what they notice when they fold the paper in half on the line and hold it up to the light.

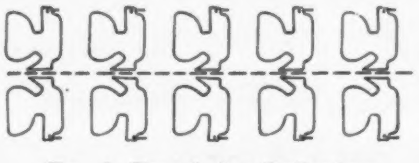


FIG. 2. Translation Reflections

3. *Two Reflections* is shown in figure 3. Draw the pattern, flip the stencil over along the right edge, and draw it again. This is your first reflection. Now flip the stencil

again along the right edge and draw the pattern. Now you have two reflections. The children can test their reflections by drawing lines between each pattern (fig. 3) and folding the two outside patterns into the middle.

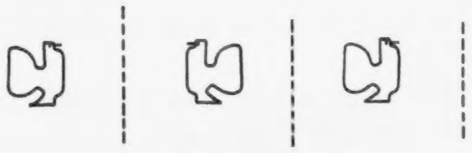


FIG. 3. Two Reflections

4. *Three Reflections* is more difficult and probably for older children only (fig. 4). For this one the children will need tempera paint. Have them fold a sheet of paper, dividing it into eight parts (fig. 5).

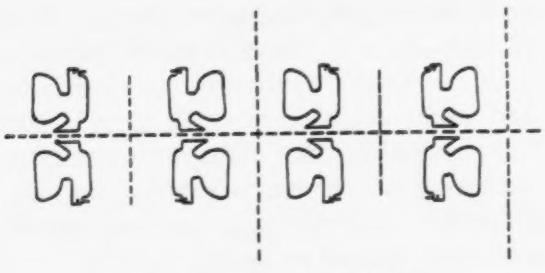


FIG. 4. Three Reflections

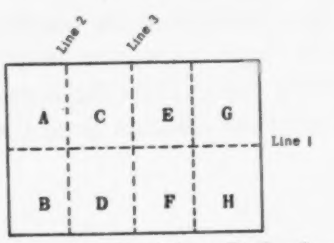


FIG. 5. Procedure for Three Reflections

Put a blob of tempera paint in the top left box (labeled "A" in fig. 5). Fold the paper in half on line 1 and press to reproduce the paint blob on box B. Unfold the paper and you have the first reflection. Now fold the paper on line 2, press, unfold, and you have the second reflection printed in boxes C and D. (If the paint is drying, apply more paint on boxes A, B, C, and D). Next fold on line 3 and press to produce the third reflection in boxes E, F, G, and H. After the children understand the operation, they can make patterns with their stencils.

5. *Half-Turns is* an easy pattern (fig. 6). Trace the stencil. Turn it 180° to the right and trace it again in this position. Move the stencil to the right while turning it 180° each time you trace it.

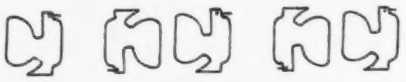


FIG. 6. Half-Turns

6. *Half-Turns about a Point* is a more difficult pattern (fig. 7). Fold a sheet of paper to divide it into six parts (fig. 8). Mark points A, B, and C as shown. Label the paper "part 1, 2, and 3" along the bottom. Use a pin to hold the stencil at point A while you trace the design in the top section of the paper. Without removing the pin, turn the stencil 180° and trace the pattern in the bottom box. Remove the stencil and cut on line 1. Holding the pin at point B, turn part 1 of the paper 180°. Lay part 2 of the paper over part 1 and trace these two patterns. Next cut on line 2 and rotate part 2 of the paper 180° at point C. Lay part 3 of the paper over this and trace these two designs.

Now tape the three parts of the paper back together as they were originally and you have a design made following the operation of half-turns about a point. Ask

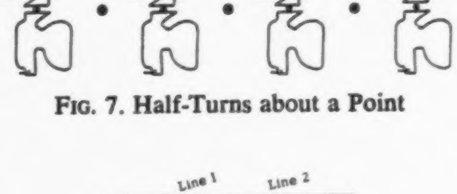


FIG. 7. Half-Turns about a Point

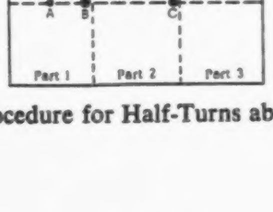


FIG. 8. Procedure for Half-Turns about a Point

the children how this design differs from the translation reflections design. At first glance the two patterns may look very similar to the children.

7. *Glide Reflections* combines the rules of moving a certain distance and reflection (flipping) the pattern along an axis (fig. 9). Draw a line on your paper to use as an axis. The pattern should be above the line (with part of it touching the line) as you trace it. Move the pattern along the line a certain distance to the right. Reflect (flip) it below the line and trace it again. Move the pattern along the line (below it) an equal distance to the right. Reflect it and trace again above the axis. Continue moving and reflecting the pattern alternately above and below the axis.

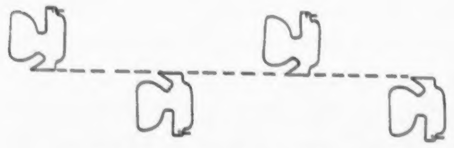


FIG. 9. Glide Reflections

After the children understand some of these basic operations, they should make up their own. There are dozens and dozens of operations that can be devised using these basic ones. This exercise will certainly bring out the creativity in your children. It also gives practice in inventing rules and carrying out procedures.

Now that the children know what to look for, they will enjoy finding examples of strip patterns in nature and art. They should be encouraged to bring all the examples they can find and to tell the class what operation was used to produce the pattern.

The children can think up games using these patterns. They might try to predict what a pattern will look like when using a certain operation. Then they can carry out that operation and see how correct they were. They can play games with partners. One child chooses the operation, and his partner produces the first step. The first child does the next step, and so on.

These strip patterns suggest many ways of decorating the classroom. Encourage the children to make things using the patterns, such as mosaics, placemats, wall hangings, and paper chain designs. They can make their patterns using different techniques, such as block, potato, and sponge printing. Have them try rubbings (crayons rubbed over paper placed on an object) with leaves and twigs. Just remind them that they must choose or invent an operation and follow the rules throughout the pattern. After the children make a few strip patterns on their own, they can explain the operation to the class or let the children guess the operation.

Developing generalizations with topological net problems

CHARLES H. D'AUGUSTINE *Ohio University, Athens, Ohio*

Professor D'Augustine is a member of the college of education at Ohio University.

Topological net "games" are being included in contemporary junior high and elementary school mathematics texts. The "game" aspect is of questionable pedagogical soundness. Wouldn't these topics be of more value to the student if they were introduced systematically and in such a way as to lead the student to make certain logical generalizations? Wouldn't these topics be of more value to the student if he "possessed" certain generalizations which would enable him to create and solve highly sophisticated net problems independently.

This article will be an attempt to show how one type of net problem (that of traversing nets) could be developed in both a logical and an intuitive way to lead to generalizations which would enable a student to create and solve "related" net problems.

This article will presuppose that students have had previous experience with end points and cross points.

All examples will be intuitively treated as follows:

Each situation will revolve around a person attempting to deliver papers to homes on a paper route. The following rules will hold with respect to delivering papers:

1 In no instance may the person delivering papers get off his bicycle; he must continue riding until he can ride no further.
2 In no instance may the person delivering papers leave the road.
3 In no instance may the person delivering papers go over a portion of a road which he has previously traveled.

With this basic set of rules we can now study specific situations of paper delivery.

Example 1. Two end point problems (Fig. 1). Joe starts delivering his papers at end point A. Can he deliver papers to the house at point M? (Yes) Where will he have to stop his bicycle? (Point B)

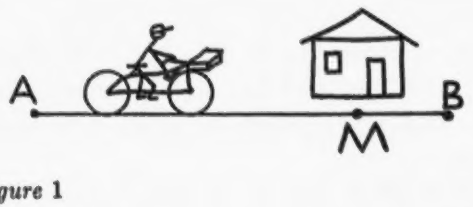


Figure 1

Example 2. Three end point problems (Fig. 2). Joe starts delivering his papers at end point U. Can he deliver papers to both houses? (No) What is the largest number of houses that Joe could deliver papers to? (One)

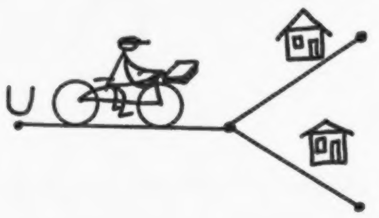


Figure 2

Generalization: I am thinking of a delivery route with 5 end points. If there is a house at each end point, could Joe deliver papers to each house? (No)

If he must start at one of the end points, what is the largest number of houses to which he could deliver papers? (Two) Why? Because as soon as he reaches another end point he must stop.

What is the largest number of end points a problem can have in order to have a solution? (Two)

To make further generalizations it will be necessary that we intuitively develop the idea of odd and even cross points.

Example 3. Joe is on one "road" (Fig. 3). When he reaches the cross point he will have a choice of two "roads" to take. Since 1+2=3 and 3 is an odd number, we say that cross point A is an odd cross point.

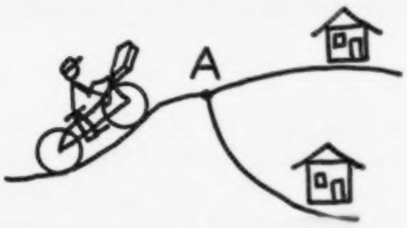


Figure 3

Example 4. Joe is on one "road" (Fig. 4). When he reaches the cross point he will have a choice of 5 different roads to take. Since 1+5=6 and 6 is an even number, we say that cross point C is an even cross point.

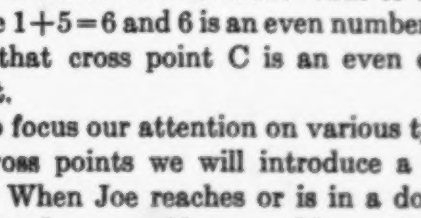


Figure 4

To focus our attention on various types of cross points we will introduce a new rule. When Joe reaches or is in a dotted portion he may ride over this portion as many times as he desires.

Example 5. Is point B in Figure 5 an even or an odd cross point? (Odd) If Joe starts in the dotted area, can he deliver papers to all three houses? (Yes) Where will his problem end? (At cross point B) If Joe starts delivering papers at cross point B, will his problem end at this point? (No! He will end up in the dotted area.)

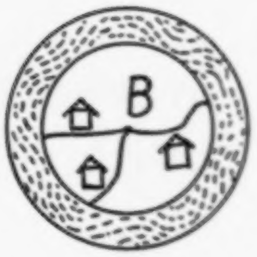


Figure 5

Example 6. Is point C in Figure 6 an even or an odd cross point? (Even) If Joe starts in the dotted area, can he deliver all the papers to all the houses? (Yes) Where will his problem end? (In the dotted area) How is this answer different from the situation with the odd cross point where Joe started in the dotted area? (When he didn't start at an odd cross point, he ended there.)

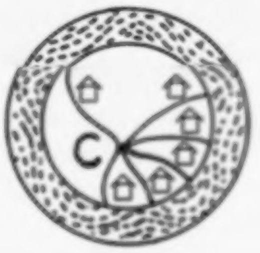


Figure 6

If Joe starts at cross point C, can he deliver papers to all the houses? (Yes) Where will his problem end? (At cross point C) How is this answer different from the situation where Joe started at an odd cross point? (He ended in the dotted area.)

Let us see what sort of generalizations we can make about even and odd cross points.

Pretend that we begin a problem at a cross point whose number is N.

We will designate the road on which we leave the cross point as 1; the next road we come back to the cross point on we will designate 2, and the next road we leave the cross point on we will designate 3, etc.

Would we be leaving or coming back on

a road designated 17? (Leaving) How do we know this? (Because 17 is an odd number, and we leave on the odd-numbered roads.)

Suppose that $N=128$. Would we be leaving or coming back on the road designated 128? (Coming back) If we start our problem by leaving from an even cross point, how will our problem end (Condition: assuming that we always get back to this cross point)? The problem will end at this cross point, because we won't have a road to leave on.

If we start a problem at an odd cross point, how will the problem end (Condition: assuming that we always get back to this cross point)? One cannot tell how the problem will end, but one can say that the problem will not end at this cross point because one will always have a road to leave on since this is an odd cross point.

We are now in a position to make a generalization concerning starting a problem at an end point.

If we start at an __________ cross point,
(even, odd)
our problem will end at this cross point, providing we can always get back to the cross point. (even)

If we start at an __________ cross point,
(even, odd)
our problem will never end at this cross point. (odd)

The question now arises what happens when we come into even and odd cross points from some other portion of the net.

With the previous analysis of starting from a cross point as a background, see if you can now arrive at the generalization that one can make with regards to coming into odd and even cross points from some other portion of the net. Do not read further until you can.

As you have probably discovered, if you do not start out on odd cross points then you must end your problem there, because as soon as you use one road to get to the cross point, then the problem becomes one of leaving from an even cross point.

Example: Assume you ride your bicycle up to a "$2N+1$" cross point. Since you used 1 road in arriving at the cross point, you have $2N$ choices of leaving the cross point.

Similarly, if you do not start at an even cross point you do not have to end your problem there, because as soon as you arrive at the cross point the problem becomes a "$2N-1$" cross point problem, and leaving from an odd cross point insures that one will not end there.

Now let us abstract the problem in the following manner. Let "e" designate an end point, "O" an odd cross point, and "E" an even cross point. (Subscripts will denote the different members with the same classification.)

Pretend that we have a single net composed of the following types of specialized points:

$$e_1 \quad E_1 \quad E_2 \quad O_1 \quad E_3$$

What point or points (based on previous generalizations) would be the most optimal place to start? (Hint: If we don't start at one of these points we would certainly end at one of these points. But if we start at one of these points we don't have to end the problem at the same point. Answer: Either the e_1 or O_1, because if we start at E_1 or E_2 or E_3, we must end the problem at E_1 or E_2 or E_3 respectively; but we also have to end the problem at both e_1 and O_1, and thus the problem would have no solution.)

If we start our problem at e_1, where would our problem end? (At O_1)

If we start our problem at O_1, where would our problem end? (At e_1)

Each problem can have only one beginning and one ending. In each of the following problems select one of the optimum beginnings and then proceed to analyze whether the problem has a solution.

1 E_1 e_1 E_2 E_3 e_2 O_1 O_2
2 E_1 e_1 E_2 E_3 e_2 E_4
3 E_1 E_2 E_3 E_4 E_5
4 O_1 O_2 O_3 E_1 E_2 e_1

(Problems 1 and 4 do not have solutions. In problem 3 you would end the problem at the even cross point from which you started.)

We are now at a point at which the student can proceed creatively. We will need to introduce the following notation: E_1^4 will mean it is a "four" cross point (that is, four roads come together to make this cross point), and the 1 will retain its previous meaning. O_1^7 will mean it is a "seven" cross point (that is, seven roads come together to make this cross point). The "e" will retain its previous designations as these use only one type of end point.

Students are now in a position to create and attack problems, such as:

Given: e_1 E_1^4 E_2^2 E_3^2 O_1^3

1 Is this a solvable problem?
2 How many different solutions does this problem have?
3 How many different bicycles could traverse this net if each bicycle must always be on a different portion of the road at the same time? (Hint: Consider how the problem must end.)
4 If the problem is not solvable, what is the greatest number of houses to which papers could be delivered?

These and many other types of problems can be attacked through the formation of generalization via logical analysis.

The placement of activities which represent terminal learnings in mathematical textbooks should be examined closely. Students should always be provided with sufficient background in a topic so that they will be in a position to explore and savor mathematics on their own. Only after such a goal is reached may we hope to make mathematics exciting and challenging for students over an extended period of time.

An Adventure in Topology—Grade 5

JEAN C. CLANCY
Scarsdale Public Schools, Scarsdale, N. Y.

TO TAKE AN ADVENTURE implies the unknown, and that is just what a group of fifth grade children did recently in mathematics. Before such a trip could be taken, the consent of all had to be obtained. In this case the pupils had a desire to learn and the teacher who acted as the guide, wished to stir the imaginations of her pupils.

In taking inventory she had to ask herself some questions. Was the present math program adequate for all pupils? Was computation the most important phase of arithmetic to be taught? Were the pupils really motivated to think in mathematical terms?

To answer the first question she would have to do some experimentation to discover just how adequate the program was, for she realized that although many pupils could handle advanced work easily, something more was needed to broaden their experience in mathematics.

As for computation, this was a skill worth developing, but not to the exclusion of others. If mathematics was completely manipulation of numbers, the electronic machines could "take over." A child's arithmetic skill and such a machine have one thing in common—both are in need of repair from time to time—but unlike the machine, a child can develop the ability to reason and use logic.

To motivate children to think in mathematical terms is a real challenge, for of all the sciences mathematics breeds a motivation that is different from that required in other academic areas. The "self-drill" needs aren't always seen in mathematics, and despite present day concerns for greater competence in math, this science has always been the most challenging group of ideas to teach students.

If the necessary skills beyond computing aren't apparent to all teachers—whatever level—how can "needs" be communicated to students? To meet the challenge of motivation, she felt an obligation to approach math in a way that was new to the pupils and herself; to begin looking for patterns of thought that have their basis in mathematics; to be able to generalize, work with theory and then make some applications that would be appropriate to the classroom situation.

The ideas took form in the beginning of the year when the fifth grade class was discussing length and width and then began to ask questions about other dimensions, and the meaning of our three dimensional world. When paper was represented as 2-D, the class was asked if they had ever seen one-sided paper. After the initial laughter and questions had subsided, they were asked to think about this and share their thoughts with classmates the next morning. Their thinking ranged from: "paper stapled to the bulletin board so only one side showed," to "a picture of a piece of paper because only one side could be photographed." Certainly there was thinking, and above all a genuine curiosity about one-sided paper.

When the teacher cut a long narrow strip of paper and pasted the ends together, in the form of a headband, the class told her that "it has two sides because we can see two sides—inside and outside." Then she took another strip of paper—longer than it was wide—and gave the paper a twist before joining the two ends together (see Figures 1 and 2). She told them this was called the Möbius Band named for a German mathematician who studied it about 100 years ago. To help them discover why this was one sided paper, she had them run their fingers down the middle of the length of the strip, following the twist carefully. After one trip around the band, they found that they had returned to the original point of departure, without having taken their fingers from the

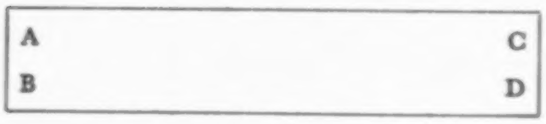


FIG. 1

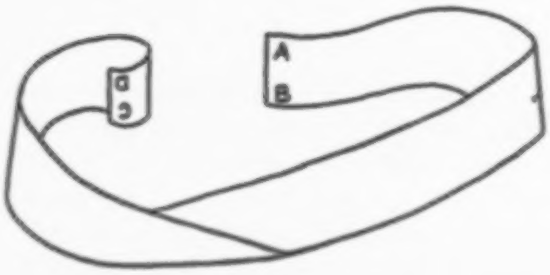


FIG. 2

paper. No edges of paper had been crossed.

When trying the same procedure with the first named ring (in the form of a headband) they discovered that there was no way to get from the inside circle to the outside circle without crossing an edge. Therefore the conclusion was that the Mobius surface was indeed an example of one-sided paper whereas the "ring" was the usual two-sided type.

Interesting to fifth graders? Yes, especially when discussion led to the three dimensional world we live in and applying what they already knew about dimensions, to our world.

The motivation for more thinking was in the new found surface before them—the Mobius Band. Yet why learn about something that had no apparent significance to them in their three dimensional world? With more thinking and reasoning taking place, some in the group ventured a "hunch" that it must have something to do with outer space. This proved to be a good idea for it was pointed out that in four-dimensional space, closed surfaces such as Mobius can exist. Mathematicians and scientists believe that it isn't at all impossible that astronomical space is closed in on itself and twisted like the Mobius Band. Amazing!

Although this would be the closest they could ever come to a Mobius Band, still it would be fun to experiment with it to see if it had any unusual properties. By cutting lengthwise through the middle, everyone was prepared to find that the band would be cut into two pieces. But somehow the unexpected happened, for instead of having two pieces, it still had one piece, but twisted twice instead of once. And worst of all it was now just an ordinary two-sided band! The fifth graders decided that this magic was all right for demonstration, but were just as glad that outer space in the form of a giant Mobius surface was free from prying scissors!

When someone recalled that you, the teacher, had mentioned four-dimensional space and "what did you mean by that because here we only have three dimensions?" it demanded all your resources to explain to ten year olds that *time* was the fourth dimension.

It didn't seem too mysterious to understand when it was explained that events that happen around us not only include distance in space but also the *time* that they happened to be completely descriptive.

Now that the measurement of space was in the thoughts of all, another word could be added to the growing ten's vocabulary—along with dimension, Mobius, surface, length, width, height. This word was Topology, the branch of geometry which means the study of locations (without reference to measurements of lengths or angles in space).

"If this is a part of geometry—the subject that my brother is studying in high school—couldn't I learn something about it too, and have sort of an adventure like we did with the Mobius Band?"

"Yes, as a matter of fact you could"—and all of us did have an adventure in geometry that led us to the very nature of things around us.

Before taking that adventure however, a final accounting was taken of what we had learned in mathematics and science as the IGY period came to its end. Of all the discoveries and experiments undertaken by scientists in all the world, our own encounter with the Mobius Surface was the most dramatic because it made us think about the possibilities of unusual properties of space still waiting to be discovered by *thinking* minds.

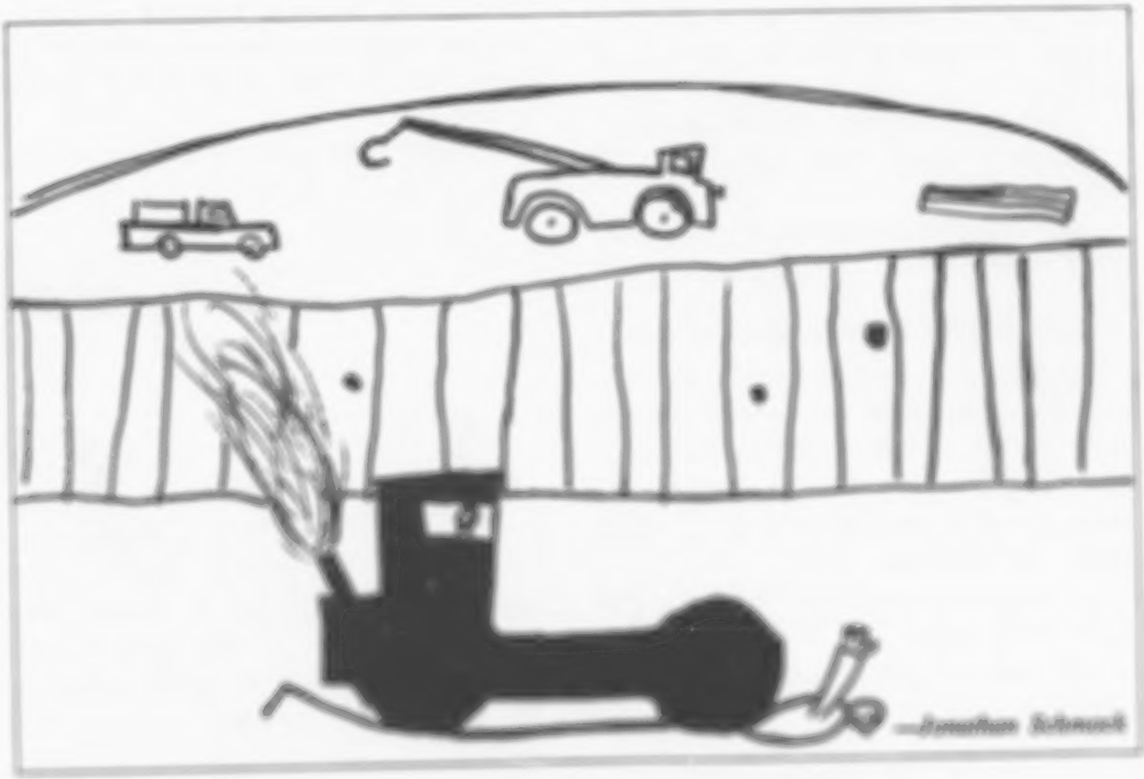
Creatamath, or—Geometric ideas inspire young writers

EMMA C. CARROL *Carroll College, Waukesha, Wisconsin*

It happened because of Flat Stanley, Jeff Brown's delightful, two-dimensional boy, who became so when a falling bulletin board flattened him out![1] My fourth graders couldn't resist this fellow whose accident cost him a dimension and gained him some adventures. Slithering under doors was fun. Being flown like a kite was even better! Fun ended, though, when the boy-kite became entangled in the treetop! Brother Arthur and his bicycle pump accomplished the feat of returning Stanley's lost dimension and making him a creature of this world once more!

"Can you think about the points, lines, and planes in your math book—and then create characters from them as author Jeff Brown did with Stanley?" an enchanted class was asked.

Sly looks, chuckles, laughing eyes, and flying pencils answered my question in less



—Jonathan [illegible]

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[1] Jeff Brown (Tomi Ungerer, Illus.), *Flat Stanley* (Evanston, Ill.: Harper & Row, 1964).

than quarter of an hour. Authors shared characters and adventures—and before the afternoon disappeared, delightful Stanley stood in the middle of the fourth-grade bulletin board in the company of twenty-four other geometric creations created by young mathematician authors!

Here are some for you to enjoy. The characters are a little like Stanley, of course—but their predicaments have original features and lost dimensions are regained in remarkable ways! Best of all, most of the chuckling authors deepened and broadened their concepts of 1D, 2D, and 3D! Do have fun with the stories which follow!

The Flat Kid, Jim, by *Pat*

One day a boy was watching a steamroller. The steamroller couldn't stop and ran over him.

"Hey! Are you all right?" asked the operator.

"I'm all right!" said Jim.

"You are flat!" said the operator. "It is scientifically impossible!"

Jim ran home as fast as he could go. "Mom! Look at me! I'm Flat Jim!"

Jim's mother was a fast thinker. She said, "Get the gasoline. Drink it, Jim."

"O.K.," said Jim, and he downed the gasoline.

"Now, get a match!" said his mother.

"Boom!" There was Jim made 3D again.

Of course, Jim never went near a steamroller again!

Linda and the Printing Press, by *Ann*

Linda was a very curious girl. She was mostly curious about her father's printing press. One day, while watching her father at his press, she got too close and the big press squashed her. As soon as the press lifted, she ran away.

An inspector came by and saw that some of the paper had prints of a girl on it. He called Linda's father and asked, "What has happened?"

"That looks like *my little girl!* She must be 2D!"

Linda's father ran home and went right to Linda's room. There he saw Linda as a *two-dimensional girl.* Everyone tried to get her back to normal, but she always stayed 2D!

—*Susan Teller*

A One-D Boy, by Bob

One day, Jim and his brother were playing with his tape recorder. Suddenly, Jim got caught and he came out 1D! He cried and cried but he could find no way to become 3D again.

It wasn't very good to become a jump rope for girls and that is exactly what happened to Jim! He was pleased when some boys rescued him and decided to use him for a kitestring.

—Connie Cross

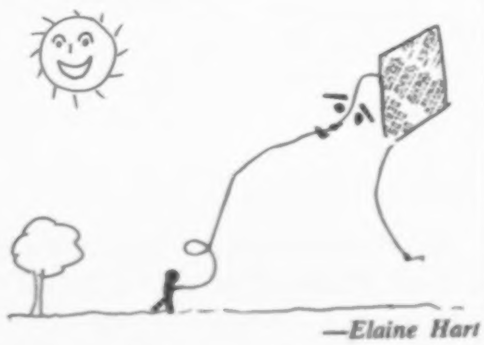
"How lucky can a boy-kitestring get?" said Jim, for as soon as he was high in the air, he opened his mouth, the *wind* sailed in and Jim became 3D again.

Steve and the Laundromat, by *Diane*

Steve and his mother went to the laundromat. Steve asked, "What would the world be like if everyone were 1D?"

His mother told him, "I guess the world would be full of strings." While he was thinking about this, he sat down on the washing and soon was hidden in the dirty clothes.

Steve's mother popped the washing into the washer and, of course, Steve went right in with the clothes. He screamed, but his mother had no idea where the scream was coming from. She shouted, "Where are you, Steve?"



—*Elaine Hart*

"I am in the washer!" answered Steve.

"I'll get you out!" said his mother. She pulled and pulled and then saw Steve, looking like a string. She cried and cried, took him out, called the doctor and cried some more.

"No bones are broken!" said the doctor. "Can he eat? Can he talk?" Steve could, so the doctor left him as a 1D string with a crying mother.

A playmate came to visit. He asked for Steve and could scarcely believe that the string who answered the door was his friend. "Yippy!" said the playmate. "We'll use you as the tail on a kite!"

Outside the two boys went, and up went Steve. He went higher and higher as the wind blew the kite into the sky. Then the friend let go of the string and no one ever saw or heard of Steve again!

EDITOR'S NOTE.—Mrs. Carrol has shown one way children may relate arithmetic to other curricular areas, in this case creative writing and literature. I suspect the children who wrote these stories not only enjoyed writing them, but also understood the meaning of one-dimensional, two-dimensional and three-dimensional figures better because of it!
—CHARLOTTE W. JUNGE.

A second example of informal geometry: milk cartons*

MARION WALTER
Harvard University, Cambridge, Massachusetts

Marion Walter is an assistant professor at the Harvard University Graduate School of Education. Her concern is the education of teachers in the mathematics program. She conducts many in-service workshops and is particularly interested in the visual aspects of learning mathematics.

This article describes some work that children can do with milk cartons. You will need paper milk cartons and construction paper. Cut the top off the milk cartons so that the height is equal to the width. Rule the construction paper with two-inch squares.

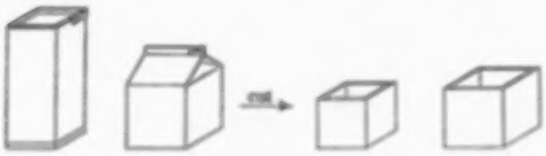


FIGURE 1

One can start the work in a variety of ways, depending on the age and interests of the children and the size of the group. I and other teachers have worked with groups as large as thirty-five and as small as five. The description given here works well with children in the third grade and above. Appropriate modifications make this an exciting unit for students from the first grade through college.

Before reading on, try some of the work yourself. Please get a couple of the cut milk cartons, some construction paper, scissors, and a friend! Ask the friend to read the next few questions and directions to you, because you won't be able to read them if you really do what I am going to ask you to do! I am going to start by asking you to close your eyes!

Visualizing the box

Close your eyes!

Visualize a box. Keep your eyes closed.
How many sides does your box have?
How big is the box you visualized?
Can you think of a bigger (or smaller) box?
Can you think of a longer (or shorter) box?
Imagine a box all of whose sides are squares.
Take the top off your box (are your eyes still closed?) so that you have a box without a top. Just to make sure that you have a good feeling for an open box, visualize filling it with sand. Pour the sand out again.
How many sides does your box have now?
Now imagine that a box manufacturer wants to ship these boxes flattened out. Can you imagine how such a box looks flattened out?

Open your eyes!

Draw how a box with five square sides looks flattened out. What did you draw?

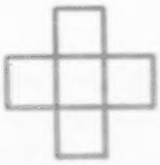
---

* For a first example see "An example of informal geometry: Mirror Cards," THE ARITHMETIC TEACHER, 13 (October 1966), 448-52.

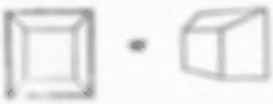
Before continuing, let me describe examples of what takes place in a classroom up to this point. When one child, visualizing a box, was asked how big his box was, he replied, "Big enough to put the whole world in." Another said, "I can hold it in my hand." Both statements could motivate a discussion of what the dimensions of such boxes might have been. Occasionally children will give actual dimensions, and when they do they tend to give only two of the dimensions. For example, one boy said, "Mine is 5 by 7," but he didn't name the unit of measurement. After challenging him, he said, "It's 5 feet by 7 feet." A discussion that involved actual measurement of boxes brought out the need for knowing three dimensions. The children also realized that one measurement was sufficient if one knew ahead of time that the box had all square sides.

Not all the children were able to recognize squares, and you may need to discuss "squareness." Many children who are quite sure that ☐ is a square are not certain when the pattern is turned.

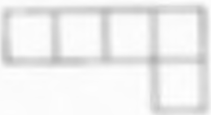
Many children draw



when asked how their box looks flattened out. Some have drawn ☐, saying "All the pieces are on top of each other." Some have tried to draw perspective drawings such as

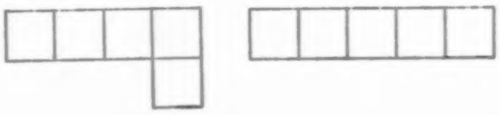


Occasionally, a student has drawn



Drawing of five-square patterns

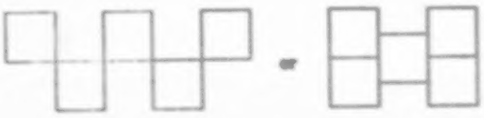
Do the two patterns pictured below fold into a box without a top?



Can you think of other patterns made of five squares, regardless of whether they fold into boxes or not?

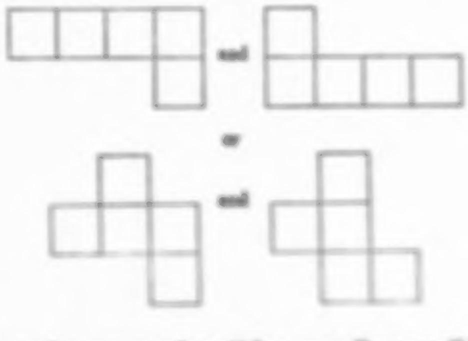
Draw as many as you can find.

What takes place in the classroom? Occasionally, some children have had difficulty finding new five-square patterns. They can cut out squares from the construction paper and use them to push around to help them form new patterns. Sometimes children may want to draw and include patterns such as



I then explain casually that for the moment we will make the rule "whole sides touching." Sometimes children investigate later how many patterns there are if the rule is "corners only touching."

Eventually, the children find all twelve patterns. The question of whether



are to be counted as "the same" or as "different" is raised by the children. They usually decide to consider two patterns to be "the same" if each of the patterns can be covered by the same paper cut-out.

Folding and unfolding

Which of the twelve patterns fold into boxes without tops?

Can you predict just by looking at each pattern

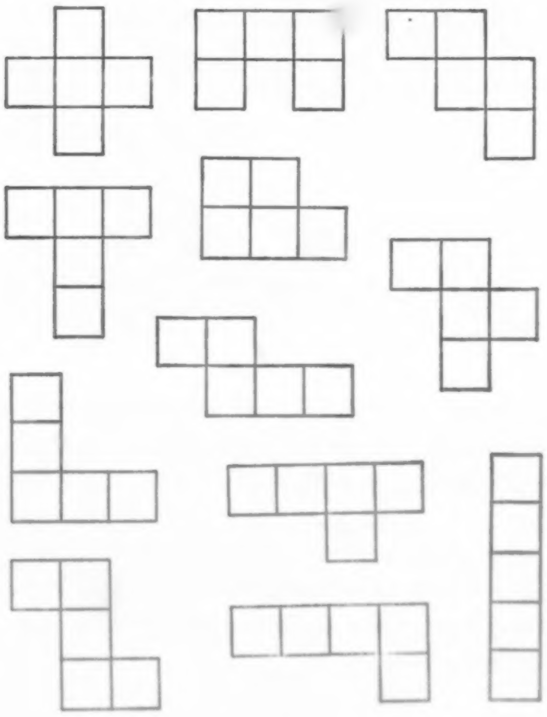


FIGURE 2

in Figure 2 which are "box-makers"? Try! Check by cutting out all the patterns and folding the paper.

Can you predict which square(s) form the bottom?

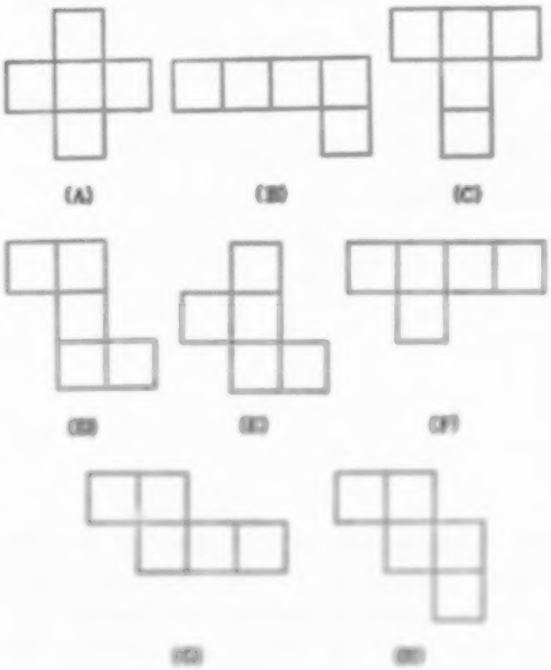


FIG. 3—The Box-makers

The children often disagree with one another when they predict which patterns are "box-makers" or which square forms the bottom. They can always settle their arguments by themselves by using the paper patterns.

Now draw all the "box-makers" and label them. See Figure 3, for example.

Choose one of the patterns [don't choose A!] and write its number on the bottom of a cut milk carton. Try to cut the milk carton to obtain this pattern!

Children much enjoy this activity and like to choose several of the patterns in succession. They are often surprised when they obtain one of the patterns but not the one they bargained for! Sometimes the cartons fall into two pieces.

Extension of the work

This work can be extended in many directions.[1] Many questions have been suggested by students themselves. Children have worked with four or six squares. They have explored equilateral triangles and solids made from them. Some investigated rectangles. The children have often tried to cut patterns with minimum wastage of construction paper.

Conclusion

This article describes only a small part of the work that can be done with patterns of squares. I chose to isolate this milk carton section because it can be done by itself without the rest of the unit and because milk cartons are being thrown away every day!

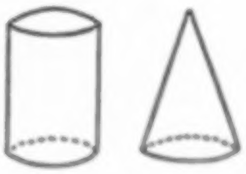
The children often do not realize that they are doing anything mathematical while working with these materials. What do you think?

---

[1] "Polyominoes, Milk Cartons and Groups." A brief description of the extended work written for high school teachers appears in the English journal *Mathematics Teaching*, No. 43 (Summer 1968), pp. 12-19.

Volume of a Cone in X-Ray

Manipulative materials: A cone made of cellulose or some transparent material that will retain its shape, a cylinder made of the same material having, of course, the same height and circumference and sawdust to be used to fill the figures.



Having learned how to find the volume of a cylinder, pupils can through the following performance proceed to an understanding of how to find the volume of a cone. Let the pupils fill the cone to the brim with sawdust and then pour it into the cylinder. Upon measuring they'll lend accuracy to their observation that only $\frac{1}{3}$ of the cylinder is filled, and through further experience pupils will see that the cylinder holds three times as much as the cone. Incidentally, it will help considerally toward clarification if the pupils mark each third of the cylinder with a color. Through questioning as well as actual performance, pupils will arrive at the conclusion that since the cone is $\frac{1}{3}$ the size of the cylinder, it will logically follow that the volume of a cone will be $\frac{1}{3}$ the volume of a cylinder, and hence the formula: $V = \frac{1}{3}\pi r^2 h$.

The advantage to be found in the use of these manipulative materials is the direct experience the pupils have in seeing as well as participating in actual solution of their problem.

Contributed by
Sister M. Vincent
Cleveland, Ohio

Geoboard geometry for preschool children

W. LIEDTKE and T. E. KIEREN
University of Alberta, Edmonton, Alberta, Canada

Thomas E. Kieren is an assistant professor of education (mathematics education) at The Pennsylvania State University, and Werner Liedtke is a graduate student in elementary mathematics education at the University of Alberta.

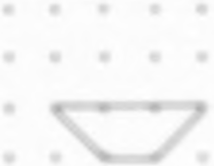
There is increasing emphasis today on preschool experiences in mathematics for children to capitalize on their eagerness to explore and interpret the world around them. This paper explores a wide variety of experiences based upon the use of a geoboard and rubber bands. The geoboard provides many opportunities to acquaint children with geometric concepts. Learning from their own play and from imitation of adults and other children, it is not long before they can recognize, correctly label, and form for themselves many common geometrical figures and instances of geometric properties, as well as such common shapes as letters of the alphabet. The illustrations that follow were drawn from the authors' observations of children aged 2–6 as they worked individually and in groups with the geoboard. Questions asked and possible suggestions given are classified under three main headings: *Familiar Shapes; Plane Figures;* and *Segments.*

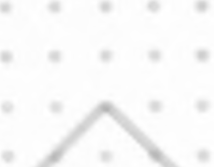
Free activity

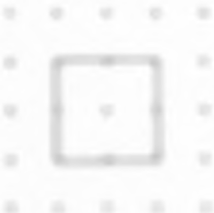
An excellent way to begin is by providing each youngster with a geoboard and a rubber band and letting him do whatever he wishes. While he works, he could be motivated to show and talk about what he has made. Recent experiences and objects from his immediate environment will be represented by various ingenious constructions and configurations. Some sample responses from a "free activity" period are presented on the following page. (Our geoboards were 5 by 5 inches, and the pegs were about 1 inch apart.) The comments recorded represent the first responses of the children. Often, slight modification or even rotation of the geoboard led to the assigning of different names to very similar figures.

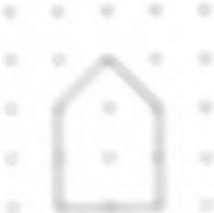
One sequence may begin by giving the children one rubber band, later increasing the number to two or even three, and challenging them to make something that was not possible before.

One rubber band:

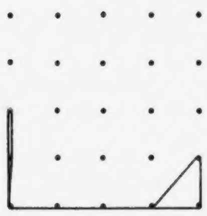

2 years
Boat

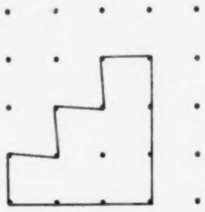

3½ years
Tent

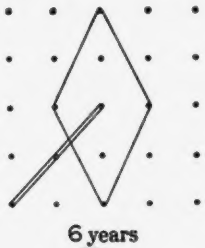

3½ years
Window

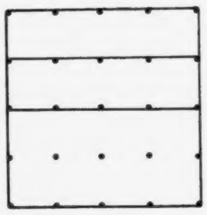

4½ years
House

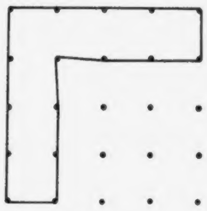
57


5 years
Bed

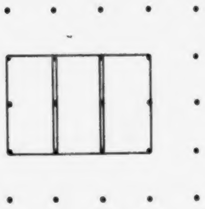

$4\frac{1}{2}$ years
Steps

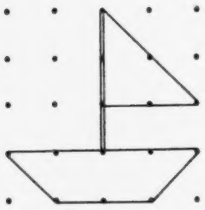

6 years
Kite


5years
"Fridge"

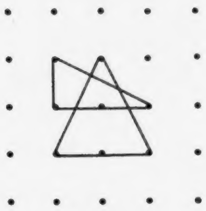

$4\frac{1}{2}$ years
Road

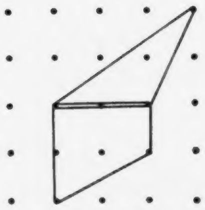
Three rubber bands:

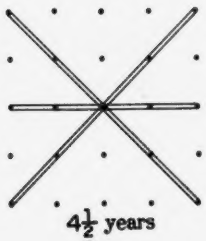

2 years
Corrals

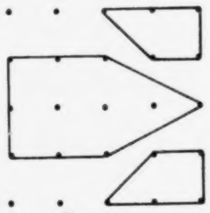

$4\frac{1}{2}$ years
Sailboat

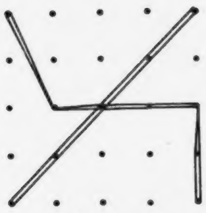
Two rubber bands:

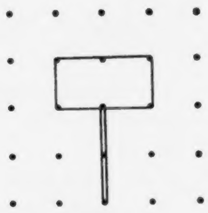

2 years
Star

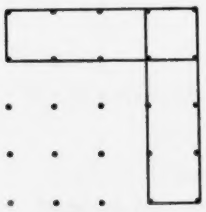

5years
Ice-cream cone

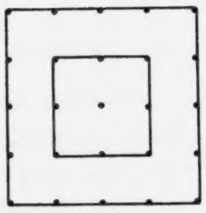

$4\frac{1}{2}$ years
Star


5 years
Bird with wings


$4\frac{1}{2}$ years
Train track and street


5 years
Canada flag


$2\frac{1}{2}$ years
Garden and sandbox


$4\frac{1}{2}$ years
Swimming pool

Familiar shapes:

SUGGESTIONS

On your geoboard, show how to make some shapes that look like something in this room.

Try to make something that can be found in the kitchen.

Try to make something that can be found in the—

basement;
yard;
grocery store;
playground;
garage.
Show something your dad uses.
Show something that is alive.

POSSIBLE QUESTIONS (INSTRUCTIONS)

Can you tell your friend what you have made? (Show him and explain.)

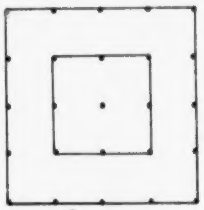
Look at something someone else has made and try to guess what it is. (Ask for a hint where it can be found.)

Does your figure look the same if you turn the geoboard around?

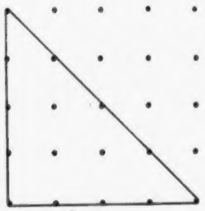
How many sides does your figure have?

How many corners does your figure have? (Are there more corners or more sides?)

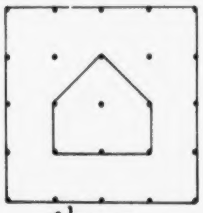
Sample responses:



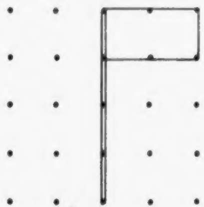
2½ years
TV



2½ years
Slide



4½ years
House and garden



4½ years
Hammer

Plane Figures:

SUGGESTIONS

Try to make figures with three sides that are—

small;
large;
"skinny";
"fat."

Try to make figures with four sides that are:

long;
short;
long and wide;
long and narrow;
short and wide;
short and narrow;
"like a square";
"not like a square."

Try to make figures with "many sides."

POSSIBLE QUESTIONS (INSTRUCTIONS)

What does the figure you have made remind you of? Does it look like anything that is familiar to you? (Where did you see something like it before?)

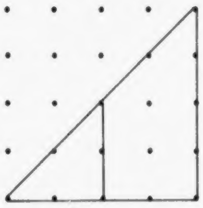
Does the figure change if you turn your geoboard?

Make two figures that: (1) do not touch; (2) touch; (3) cut into each other. Look at the figures you have made.

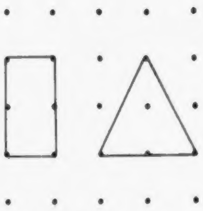
Can you make another one that looks just like it—but smaller, (or bigger)?

Make a triangle and a square. How are they alike? How are they different?

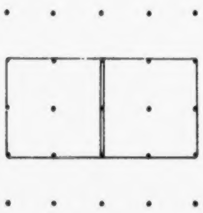
Sample responses:



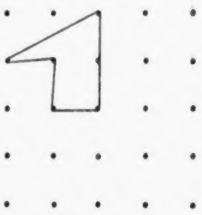
4½ years
(Make another
triangle like it,
but bigger)



5 years
(Two figures that
are different)



6 years
(Two figures that
are alike)



5 years
(Figure with many
sides)

Segments:

SUGGESTIONS

Try to make segments that are—

short;
long;
straight;
"crooked."

Try to make segments that—

do not touch;
touch;
cross each other (intersect);
will "never" touch (parallel);
are exactly on top of each other.

Try to make various segments—

leading to two (or more) points;
various numbers of segments, i.e., two that are equal;
two that are not equal;
many different segments.

POSSIBLE QUESTIONS (INSTRUCTIONS)

How would you make a road?

Can you make a very narrow road?

Can you make one that is long and narrow?

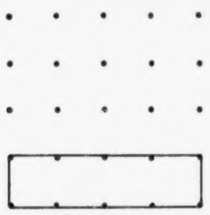
Make a railroad track. (If possible provide rubber bands of different colors.)

Can you make a road and a train track that cross? . . . do not cross? . . . will never cross?

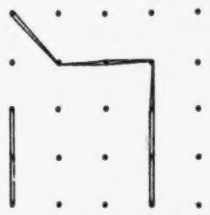
Look at two pegs in different corners. How many different roads—crooked or straight; few or many corners—can you build between these two pegs?

Which road would you like to travel on? Why?

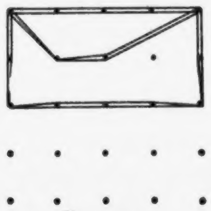
Sample responses:



2½ years
(Road)



4 years
(Long and short road, not crossing)

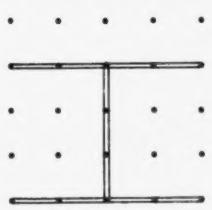


5 years
(Many roads between two towns)

Summary and additional suggestions

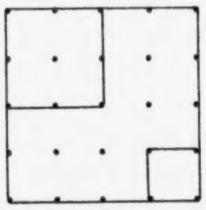
The previously outlined activities present one possible way to begin a session with a geoboard. Since the children work with creations on their own that differ in many respects, the activities are open-ended. It will soon become evident that any session will be a combination of what has been suggested. Depending on the age and background of the children, they will interpret instructions and questions in various ways. They will give unique replies that often lead to some idea that was not intended at the outset. While working, some children will recognize configurations that suddenly remind them of something familiar. For example, while talking about roads and train tracks (segments), one girl looked at her constructed figure and remarked, "A 'T' on a line!" The question was raised, "Does it look like a 'T' if you turn your geoboard around?" "No, now it's an 'H'." The question "Can you make other letters on your geoboard?" resulted in having two children make the following:

I, D, L, M, N, 4.

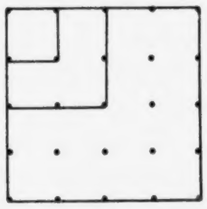


4½ years

The last response led to an attempt to build more numerals. Thus the topic of segments led to letters, numerals, and sets, and it could have also been used to discover something about angles (i.e., right angles).



2 years
(Big square—little square)



2 years
(The square that grew and grew)

Similarly, the topic of "big and little" figures can lead to the discovery of some of the properties for similar figures (corresponding sides and vertices). Having children attempt to copy a figure can lead to discovery of some properties of congruence. Some children will exhibit an awareness of symmetry (e.g., "birds with wings"). Most of them will easily pick up such terms as triangle, square, and rectangle and use these terms correctly. Some will talk of polygons and angles, and a few might even be led to discover such polygons as parallelograms, quadrilaterals, or trapezoids.

In allowing for these developments it should be remembered that these children develop ideas through both imitation and free play. By imitating an adult or another child, the child may get an idea he never had before. But free play allows him to expand on the ideas and capabilities that he already possesses.

Pegboard geometry

LEWIS B. SMITH *University of Wisconsin, Madison, Wisconsin*

Mr. Smith is a graduate student at the University of Wisconsin, where he is majoring in elementary school mathematics. He has taught and served as an administrator at the elementary school level.

Geometry has now won for itself a place in the elementary school mathematics curriculum. Although emphasis varies from publisher to publisher, there is no question that geometry is included in each major series. The Cambridge Report urges a more distinguished place for it in the curriculum of the future.

Geometry finds its place because of the contribution it makes to children's thinking. Children are led to reason, to examine geometric forms carefully, and to develop with a minimum of formal definition a skill in seeing and speaking about conditions in various parts of a geometric construction. Clarity of thought, exactness of expression and a greater feeling of self-dependency, trust in one's own ability to perceive, to examine, to hypothesize, and to prove—all assure geometry of its place in today's and tomorrow's mathematics curriculum for the elementary school.

Professional mathematicians who suggest geometry topics for the elementary school call for giving children the opportunity to use mathematical descriptions of external reality; to sense that geometric topics are intrinsically worthy of interest and respect; to interpret and compose satisfactory models involving spatial ideas; to be afforded a source of visualization for arithmetical and algebraic ideas; to discover the meaning of symmetry, equality, inequality, congruence, and similarity; and to test the appropriateness, truth, and pertinence of information.[1]

The experiences reported here are fully in keeping with the spirit and vision of these proposals. Intermediate grade children responded to the following activities with enthusiasm, understanding, and interest.

Children were given 10″×10″ pegboards. They were asked to form a rectangle 2″×3″ with four pegs and enclose the pegs with a rubber band. Then they were asked to double the width of the figure. The children set 2 new pegs and increased the stretch of the band to encompass the new figure. They were asked how the perimeter and area were affected by the change (see Fig. 1*a*). The children were asked to return the band to the original four pegs and double the length by placing two new pegs and stretching the band to enclose the new figure. Again, they were asked what effect this had on both the area and perimeter (Fig. 1*b*). Finally, the children were asked to double both the length and width of the original figure. In response to this request instances like that shown in Figure 1*c* appeared, and these offered opportunity for discussion. When Figure 1*d* was agreed upon, questions of perimeter and area were again raised.

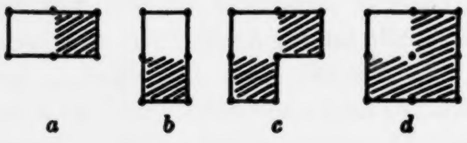


Figure 1

The expanded rectangle in Figure 1*d* was used to extend experiences. Children were asked to show ways of dividing the figure into two equal-sized areas. Early replies were confined to halving by banding off equal, rectangular areas (Fig. 2). Illustrations here are limited to one phase, although in practice the children also noted the opposite or congruent phase. Equal areas with varying perimeters were noted.

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[1] *Goals for School Mathematics*, the report of the Cambridge Conference on School Mathematics (Boston: Houghton Mifflin Company, 1963).

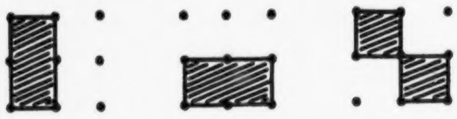
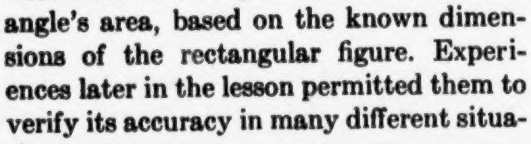


Figure 2

Other replies were forthcoming in response to the idea that they halve the original area, and much thought and discussion were generated over the equivalence of areas described. Figures 3*a* and 3*b* were quickly agreed to, but 3*c* and 3*d* were judged correct only after construction and study of the complements.



a *b* *c* *d*

Figure 3

Opportunity arose for discussion of the area of the triangles, their various shapes, and similarities of their areas. Children were able to devise a formula for a triangle's area, based on the known dimensions of the rectangular figure. Experiences later in the lesson permitted them to verify its accuracy in many different situations.

When the children understood triangular regions, many smaller triangles appeared on their boards, although the descriptions always represented half the marked-off area (Fig. 4).

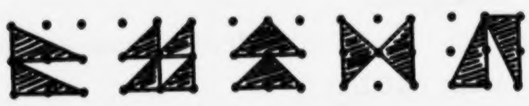


Figure 4

Then came the more intricate figures composed by the fifth graders (Fig. 5). They occasionally composed the shaded areas by assembling smaller triangles with several bands. Other children stretched a single band to encompass the largest area the band could enclose.

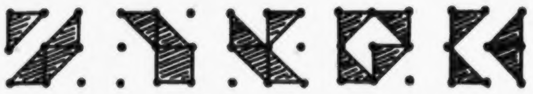


Figure 5

Finally, patterns developed which caused so much discussion and puzzlement that manila graph paper was used to help verify the similarity of areas (Fig. 6). Cutting the line-graph area proved helpful in some instances.

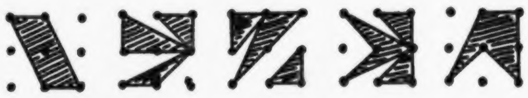


Figure 6

Additional sophistication in communicating about diagrams can be encouraged by labeling the pegs with letter names and having the children describe their patterns in terms of the letter names. Children gained accuracy in the description of areas enclosed by the rubber bands by naming the pegs. Figure 7 was described as four triangles, $\triangle AEF$, $\triangle EFG$, $\triangle BCF$, and $\triangle FHI$, by children who had used four rubber bands. However, other children preferred to name the area as a quadrilateral $AEGF$ plus $\triangle BCF$ and $\triangle FHI$.

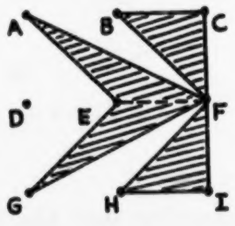


Figure 7

Another experience involved the use of four pegs. The children were asked to form as many triangles as they could by using three of the pegs.

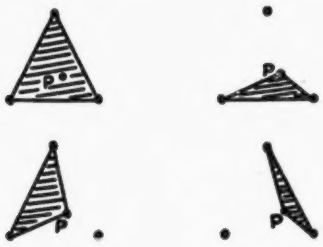


Figure 8

We then moved peg P, as in Figure 8, to location P' in Figure 9 and tried again.

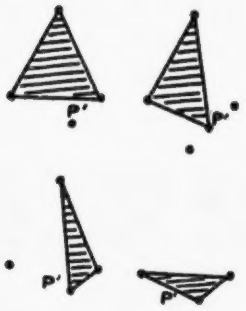
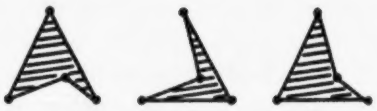
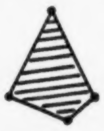


Figure 9

Children were then asked if they could place the band on the pegs in such a way that they could make four-sided figures (quadrilaterals). They discovered these figures.



(Using Figure 8 model)



(Using Figure 9)

Figure 10

On a succeeding day the fifth graders examined the triangle with a base of 2 and a height of 8 peghole intervals (inches). They quickly found triangles of equal area like the ones in Figure 11.

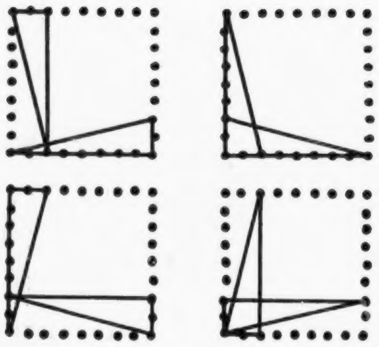


Figure 11

Questions arose when we considered triangles such as those depicted in Figures 12 and 13. These triangles offered further possibilities for discovery on the part of the children.

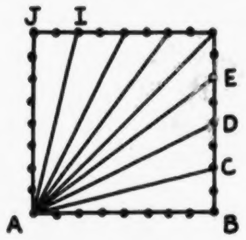


Figure 12

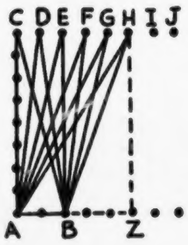


Figure 13

In Figure 12 children easily found the area $\triangle ABC$ and $\triangle AIJ$ based on the experiences described earlier. Since peghole gaps are one inch each, the remaining triangular areas were computed by finding the area of adjacent triangles and subtracting this from the enlarged triangle. For example:

Area $\triangle ACD$

$=$ area $\triangle ABD$ minus area $\triangle ABC$

Area $\triangle ACD$

$=(\frac{1}{2}\cdot 8\cdot 4)$ minus $(\frac{1}{2}\cdot 8\cdot 2)$

Area $\triangle ACD$

$=8$ square inches.

Similar treatment in Figure 13 yielded accurate results. The area of $\triangle ABH$ $=$ area $\triangle AZH$ minus area $\triangle BZH$. (Twenty square inches minus 12 square inches equals 8 square inches.)

We then explored with children what happens when the $\triangle ABH$ is extended indefinitely to points I, J, and beyond. Several children were particularly fascinated with this discovery.

Another opportunity for analysis lies in the examination of relationships of $\overline{AB}$ to $\overline{AZ}$ and $\triangle ABH$ to $\triangle AZH$ as in

$$\frac{\triangle ABH}{\triangle AZH}=\frac{\overline{AB}}{\overline{AZ}},$$

and with substitution

$$\frac{\triangle ABH}{20}=\frac{2}{5}$$

$$5\cdot\triangle ABH=40 \qquad \therefore\ \triangle ABH=8.$$

With, and sometimes without, encouragement children will ask about the equality of areas in $\triangle ABC$ and $\triangle DEF$ in Figure 14.

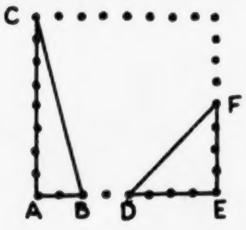


Figure 14

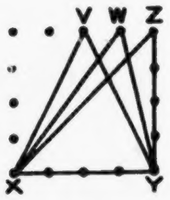


Figure 15

By this time the children's inquiry led to Figure 15 (enlarged part of Figure 14) and analysis of equality of area for numerous triangles such as $\triangle XYZ$ and $\triangle XYW$ and $\triangle XYV$, and so forth.

Experiences described in Figures 11 through 15 served to extend children's insight. The formula for a triangle's area would have sufficed to solve problems raised in these situations. However, the author feels that many discoveries and learnings await the child whose interpretations are not limited too early by the formula.

Developing a sense of trust in one's own ability to perceive and an increased awareness of the varied choices or responses available to the thinking person are a few of the rewards awaiting the elementary school student of geometry. The methods reported here amply provide these kinds of experiences. They also develop the pupils' ability to interpret physical situations through the construction of simple models. The pegboard is a successful medium for portraying geometric ideas and for promoting exploration. Through its use boys and girls can be guided to discover truths that reveal significant arithmetical and geometric understandings.

Tinkertoy geometry

PAULINE L. RICHARDS
Bethesda Elementary School, Bethesda, Maryland

Mrs. Richards' article grew out of a presentation that she made in a course at the University of Maryland. She is a classroom teacher.

Finding suitable instructional aids for my fifth-grade geometry class was a problem until I discovered the versatility of Tinkertoys.

The round "joints," or disks, though large, may be used as a representation of a point (Fig. 1).

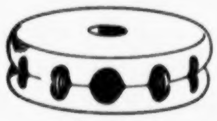


FIGURE 1

A model of a line segment can be formed by joining short sticks and more disks (points), as shown in Figure 2.

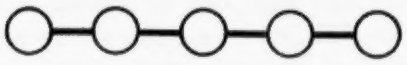


FIGURE 2

The two end points and some of the points between are clearly represented, so the property of "betweenness" can be pointed out by using this model.

If a cardboard arrow is attached, one end point is eliminated and a ray is shown (Fig. 3A). If an arrow is used to represent an extension in the opposite direction, a line is illustrated (Fig. 3B).

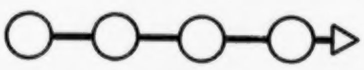


FIGURE 3A

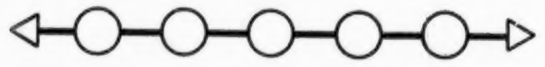


FIGURE 3B

These representations can be easily manipulated. Geometric figures can be shown as being a part of different planes rather than as being just on the surface of a chalkboard or a textbook page.

One of the balls that are now included in the kits can be used to illustrate a point through which many lines pass (Fig. 4).

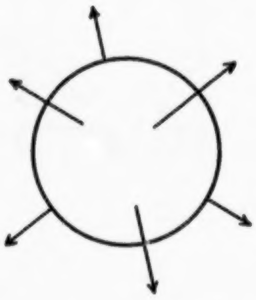


FIG. 4.—Lines pass through the point in an endless number of directions.

Intersecting lines can also be shown, and the angles can be observed (Fig. 5).

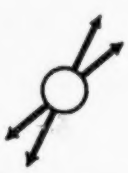


FIGURE 5

Angles can be compared by matching one representation with another and asking, "Are they congruent?" and, "Is the measure of one greater than the measure of the other?"

As the study of angles is continued, two rays with a common end point can be shown in a variety of ways. Names can then be given to the different classes of angles observed, as shown in Figure 6.

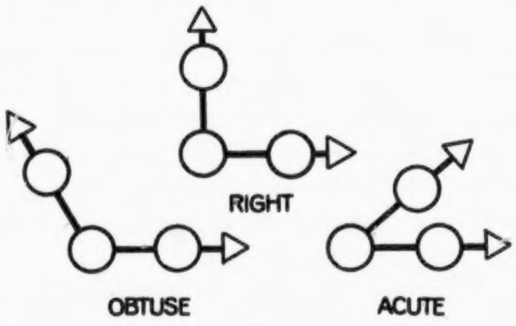


FIGURE 6

By the use of three intersecting line segments, triangles can be introduced. I encourage the students to make Tinkertoy triangles, to include angles of different sizes, and to note the effect on the length of the sides of the triangles. The students soon want to know how to name triangles, and they quickly begin discussing scalene, isosceles, and equilateral triangles.

The study of polygons continues with much interest. By making representations of polygons and using only four sticks but a variety of lengths, students readily see that quadrilaterals are not limited in shape to the familiar squares and rectangles. We are off again, naming quadrilaterals! (Actually, I had not planned to go into the naming of polygons so completely with my fifth-grade class, but I found myself carried along by their inquisitiveness.)

Illustrating a figure as the union of disjoint sets of points becomes a natural!

The representation of a line can be separated into the three sets of points illustrated in Figure 7.

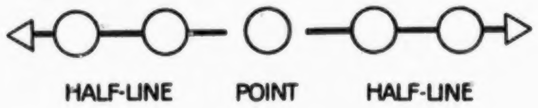


FIGURE 7

A plane may be separated into three disjoint sets of points. For instance, by using a model of a rectangle in a plane, it is possible to separate the plane into three disjoint sets of points, as indicated in Figure 8.

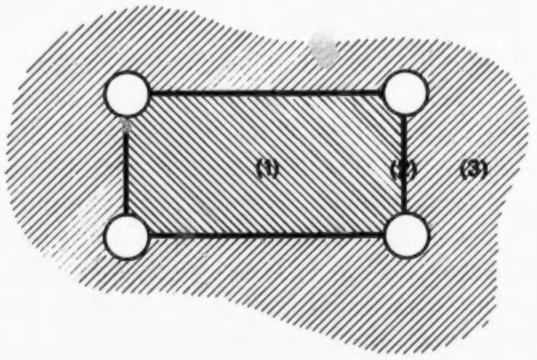


FIG. 8.—(1) indicates the points in the interior of the rectangle; (2), the points of the rectangle; and (3), the points in the exterior of the rectangle.

By passing an object through the interior of the rectangle, a distinction can be made between a rectangle and a rectangular region.

Geometric figures constructed from Tinkertoys are light enough to be used for a bulletin-board display. A small roll of masking tape on the back of each disk, and a pin through the hole, will hold any figure secure.

I am still discovering uses for my Tinkertoys—concave and convex figures and diagonals of polygons are but a few possibilities. Perhaps you can continue to discover, too!

Congruence and measurement

STANLEY B. JACKSON
University of Maryland, College Park, Maryland

Dr. Jackson is a professor of mathematics at the University of Maryland.

The elementary ideas in mathematics are abstractions drawn from the experiences an individual has with his physical environment. These experiences are primarily of two kinds. There are, first, those experiences that are associated with counting. They arise naturally in dealing with collections of discrete objects and lead ultimately to the abstract notion of the counting numbers and the operations on the counting numbers. That is, the experiences of this type are those that induce us to invent the counting numbers and the usual arithmetic of this system and its extensions. The second category of experiences includes those of a spatial nature. They are related to our perceptions of size, shape, and form. The child discovers that this pencil is too long to fit in that box; that this peg will just fit in that hole; that the faces of two of his blocks fit exactly on each other, and so on. These are the experiences that motivate us to identify and discuss various figures and to give names to the common ones like line segment, line, ray, plane, triangle, circle, and sphere. Thus the developing space perception of the individual leads to the creation of the mathematical ideas normally associated with the word "geometry."

Both these strains of mathematical thinking occur in the elementary school mathematics curriculum, with the larger portion of time devoted to the arithmetic work for which it is necessary to develop not only the concepts but also the appropriate manipulative skills. The geometric phases of the curriculum are in large part related to the idea of measurement. This is wholly appropriate, but not infrequently the emphasis is placed so much on computational skills that the essential geometric ideas involved are not fully perceived. It is the purpose here to explore briefly the geometric ideas that underlie the measurement process.

It should be noted at the outset that the subject of geometry as it occurs in the elementary school is quite different from the corresponding subject in high school. In the high school development the emphasis is quite strongly on the deductive method, with discussion of axioms, theorems, how to give a proof, and the like. At the elementary school level, while the ability to reason is to be encouraged in all possible ways, geometry is much more a development of perception, an exploring of spatial relationships. It might perhaps be described as physical geometry, and is the intuitive basis against which the later, more formal work can make sense. There will be much room for intuition and discovery. A word of caution may be in order on this point, however. Care should be taken to avoid giving the pupil the impression that a few physical observations constitute a proof. They may form the basis of a hunch or a conjecture, but to pass this off as a proof is to precipitate great confusion later when proofs are to be discussed.

The idea of congruence

In the sense of developing spatial perceptions, it is clear that geometry begins well below the school level. The child will begin early to distinguish between such things as a round object and a square or triangular one. An imaginative first-grade teacher of my acquaintance makes a deliberate effort to reinforce this discrimination by placing a number of objects of different shapes in a bag. She then has a pupil put his hand in the bag and describe, on the basis of feeling alone, the characteristics of one of the objects. Another activity in which children frequently engage is the assembling of puzzles. Indeed, the assembling of jigsaw puzzles as a recreational activity is by no means confined to children. The individual, of whatever age, who correctly selects the piece to fit into a given space in a puzzle is exercising his perception of an extremely important geometric relation called *congruence*. That is, he is observing that a particular piece of puzzle exactly fits in a particular hole. The author knew one student whose perception of shape was so keen that he preferred to assemble a jigsaw puzzle with the pieces turned wrong side up so that he was not "distracted" by the picture.

In general we speak of two figures as being congruent if one will fit exactly on the other, or, more precisely, if a model of one will exactly fit on the other. This concept of congruence proves to be a critical one for the understanding of measurement. Once the idea has been identified, it can be noted again and again, both in common, everyday contexts and in guided experiences designed for the purpose. For example, two sheets of paper from the same ream provide an excellent representation of congruent rectangular regions, and two soda straws are a reasonably good representation of congruent line segments. Even in situations where congruence does not appear explicitly, it is often in the background. If we try to compare two pencils by laying them side by side on a desk with their erasers together, it may happen, as in Figure 1, that the tips do not match. That is, the pencils cannot be considered as representing congruent segments. However, the very process of laying them side by side suggests that one of them is congruent to a portion of the other. Thus the process of observing that two segments are not congruent amounts to identifying one of them as congruent to a part of the other.

FIGURE 1

Experiences with congruence

An observation concerning congruence of angles may be obtained by using an ordinary sheet of paper. If one tears off two corners from the sheet of paper, the torn pieces will almost certainly not be congruent, yet by placing one on the other it appears (Fig. 2) that they can be so placed as to "fit" near the vertices. This leads in a natural way to the concept of angles' being congruent even when the regions with which they are associated are not. In particular, the four angles associated with the corners of an ordinary sheet of paper appear to be congruent. Moreover, if the four corners are torn off, it is discovered that these four pieces appear to just fit together at a point, as indicated in Figure 3. This property is characteristic of what is called a *right angle*. That is, an angle is a right angle if four congruent copies of the angle and its interior will just fit together to

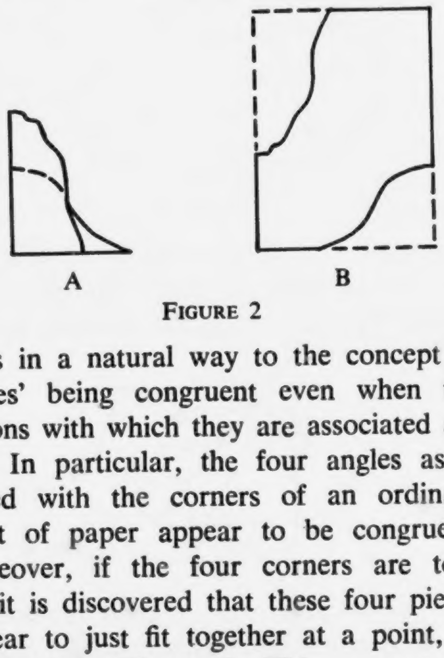


FIGURE 2

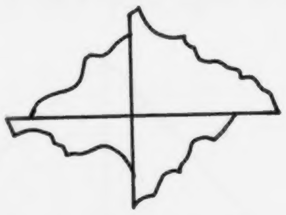


FIGURE 3

cover the space about a point in a plane. This leads to the possibility of making a very satisfactory model of a right angle by paper folding as follows. Let a sheet of paper (which is not assumed to have any particular regular shape) be folded, creasing firmly. The crease will be an excellent representation of a line segment, as indicated in Figure 4.

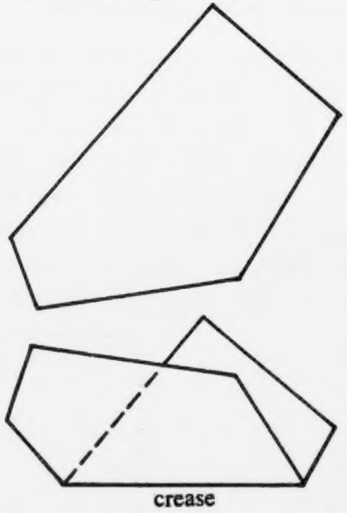


FIGURE 4

A second fold is then made so that the first crease falls on itself (as shown in Figure 5), and is again creased firmly.

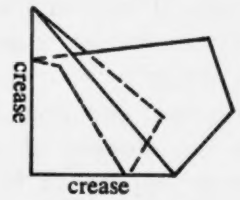


FIGURE 5

The two creases form a good representation of a right angle. This becomes clear if the paper is unfolded again. The creases are then seen (Fig. 6) to form four angles that fit together at a point and yet are congruent, since they lie on each other when the paper is folded.

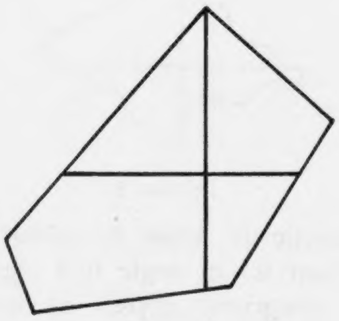


FIGURE 6

A second property of congruence which lends itself to physical verification concerns isosceles triangles, that is, triangles which have some pair of sides congruent. Consider the isosceles triangle *ABC* shown in Figure 7. We suppose that segments *AB* and *BC* are congruent. Imagine that such a triangular region is cut from paper. If the paper is folded so that vertex *A* falls on vertex *C* and if the paper is creased, the situation is as shown in Figure 8. That is, the angle with vertex *A* is congruent to (will just fit on) the angle with vertex *C*. This is an experimental discovery of a basic property of isosceles triangles which, at a later stage, the student will see as a theorem in geometry. The theorem could be stated in some such form as the following:

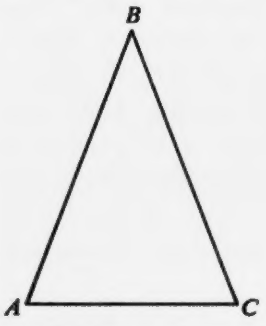


FIGURE 7

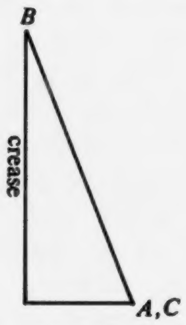


FIGURE 8

If two sides of a triangle are congruent, the angles whose vertices are opposite these sides are also congruent.

Numerous other situations involving the concept of congruence could be developed, but one more will suffice here. Imagine narrow strips of paper which can be connected at their ends by fasteners. The strips are crude representations of line segments. If three of these are fastened together, as shown in Figure 9, they form a model of a triangle.

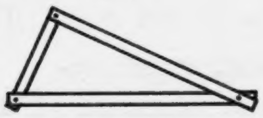


FIGURE 9

Now suppose that several persons are each given three such strips so that each two people have strips that are congruent (just alike), and suppose that each person proceeds to form a triangle as above. It will then be found that all the resulting triangles appear congruent to each other. That is, in a certain sense there is only one way of forming a triangle whose sides are congruent to three given segments. This is the physical indication for the theorem that if three sides of one triangle are congruent to the three sides of a second, then the triangles must be congruent.

The experiment above can be made even more striking by considering the corresponding situation with four strips, which can be assembled to form a quadrilateral. Here experimentation readily shows that it is *not* true that congruence of the sides will insure congruence of the quadrilaterals. An illustration of this is shown in Figure 10.

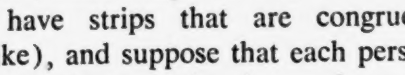


FIGURE 10

The difference between the cases of the triangle and quadrilateral is particularly impressive if the experiment is done with strips of wood or metal rather than paper. The feeling of rigidity for the triangle is in sharp contrast to the easy distortion of the quadrilateral. Pupils often find it interesting to look around for examples where the rigidity of the triangle is used in building as, for instance, a brace for a shelf or a diagonal reinforcement for a gate in a picket fence.

Linear measurement

So far our discussion has concerned itself only with the intuitive meaning of congruence. What relation does this have to measurement? What, indeed, do we mean by "measuring" something, say a line segment? The idea is to make use of numbers to describe in some sense "how much" line segment is present. The whole numbers (i.e., the counting numbers together with the number zero), as we have noticed, grow out of our experiences in counting and at first sight seem wholly inappropriate for working with an entity such as the line segment shown in Figure 11. What is there here to count? One is reminded of the frustrated golfer who bitterly described golf as a game in which the object is "to propel a little round ball from one place to another with instruments singularly ill adapted for the purpose." There is a sense in which the whole numbers are "singularly ill adapted" for the purposes of measurement. However, the techniques by which numbers are attached to geometric objects are among the most far-reaching ideas in all elementary mathematics. We meet here, so to speak, the wedding of arithmetic and geometry.

FIGURE 11

As has just been noted, when one looks at a segment, he sees nothing to count and hence at first sees no way of describing it by a number. In a sense his first task is to create something to count. This is done, as the reader is well aware, by selecting some convenient segment as a unit. Thus, in Figure 12, to measure segment *AB*, using segment *PQ* as a unit, one asks how many congruent copies of *PQ* are required to cover segment *AB*. Using segment *PQ* as a unit, it appears that *AB* is 8 units long.

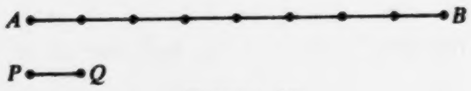


FIGURE 12

Seen in this perspective, it becomes clear that congruence of segments is at the very heart of measurement of segments. Congruence is the concept that has been used in obtaining the objects we count. To measure a segment, then, is to count the number of (nonoverlapping) congruent copies of the unit segment necessary for covering. Two segments on a line are called nonoverlapping if they have no interior points in common, i.e., if they either have no points in common or have a common end point. As we shall note briefly below, the same essential idea of measurement applies also to angles and to plane and solid regions.

While it would take too much space to discuss in detail all the consequences of this idea of linear measurement, it is possible to mention some of the ideas that grow out of it. First of all, any experience with linear measure makes clear that the process is inherently approximate. Most of the time, when one seeks to cover one segment with congruent copies of a second, it does not "come out even." For example, in Figure 13 the length of segment *CD* using *XY* as a unit is more than 3 but less than 4 units. Thus in terms of a given unit it is clearly not always possible to cover exactly with any whole number of units, and this would still be true even if we could lay off the congruent copies of *XY* without any manipulative inaccuracies. The concept of the approximate nature of measurement is an important one.

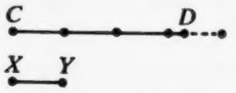


FIGURE 13

A second by-product of the idea of measurement of segments arises by observing that, in laying off successive copies of the unit segment, it might be worthwhile to mark the end point of each copy to indicate how many times the unit segment has been used. Thus, in Figure 14, point *B* is marked with a 5 to indicate it is the end of the fifth congruent copy starting at *A*. If we imagine the process to continue indefinitely, we have precisely the familiar number line. Presumably we would wish to associate the starting point *A* with the number zero. A movable copy of a part of such a number line is nothing but the familiar ruler. It is to be noted that the unit may be any segment whatever. There is no requirement in any of the discussion that a standard unit be used. The only reason for using a standard unit would be to facilitate communication with other people.

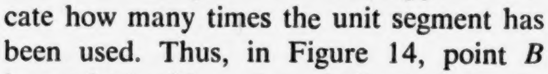


FIGURE 14

In considering the approximate nature of measurement, it inevitably occurs to one to abandon the requirement that the measure be a whole number and allow fractions. For example, segment *CD* in Figure 13 may appear to have a length of 3⅓ units or $\frac{10}{3}$ units. This is certainly a legitimate idea and leads at once to the concept of the number line with numerical labels for points at various fractional parts of units. A little reflection reveals, however, that the situation is not basically changed. To say that *CD* has a length of $\frac{10}{3}$ units means simply that we have considered the original unit segment divided into three congruent parts. The statement then says that it takes 10 congruent copies of this smaller segment to cover *CD*. Thus the use of fractional measures, while frequently useful and desirable, really amounts to nothing more than selecting a new and smaller unit in which to measure.

Angle measurement

The essential idea of the measure of a segment as the number of congruent nonoverlapping copies of a unit necessary for covering extends readily to angles. Consider the angle *AOB* in Figure 15, and let angle *RST* be selected as a unit angle. Then by the measure of angle *AOB* with respect to this unit we mean the number of nonoverlapping congruent copies of angle *RST*

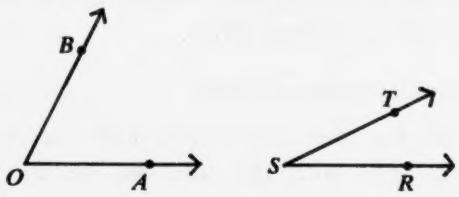


FIGURE 15

and its interior that are necessary to cover angle *AOB* and its interior. The word "nonoverlapping" means that successive copies have a ray in common, but that their interiors have no common points. Showing the successive congruent copies by the dotted rays in Figure 16, it appears that angle *AOB* is between 2 and 3 units. That is, it is larger than an angle whose measure is 2 and smaller than an angle whose measure is 3.

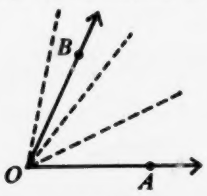


FIGURE 16

Just as the operation of measurement of segments leads to the invention of the ruler as a measuring device, so the operation of measuring angles leads to the invention of the protractor as indicated in Figure 17. Here angle *LMN* has a measure of 3 units. In this particular example, the unit angle is such that eight congruent copies of the unit angle and its interior just cover a line and one of the half planes determined by it. A more common unit of measure for angles, the degree, is such that it takes 180 congruent copies to cover the same set of points.

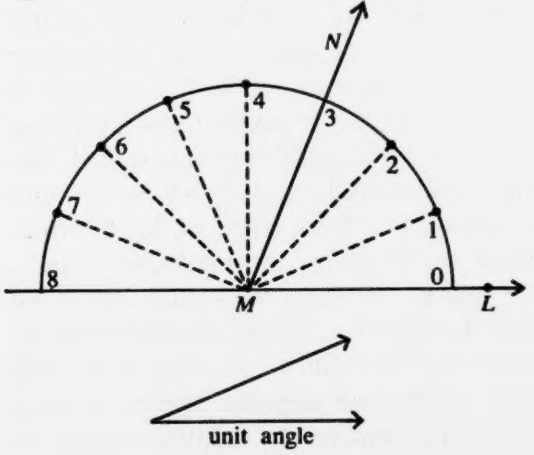


FIGURE 17

Area measure

Let us turn now to the question of measure for a region in a plane. By the word "region" is meant a simple closed curve in a plane and its interior. The essential idea is the same as before. We select some convenient region to serve as a unit and ask how many nonoverlapping congruent copies of this unit region are necessary to cover the given region. In this case, however, we find ourselves confronted with a bewildering variety of possible shapes of regions and possible choices of unit. Three possible examples are shown in Figure 18.

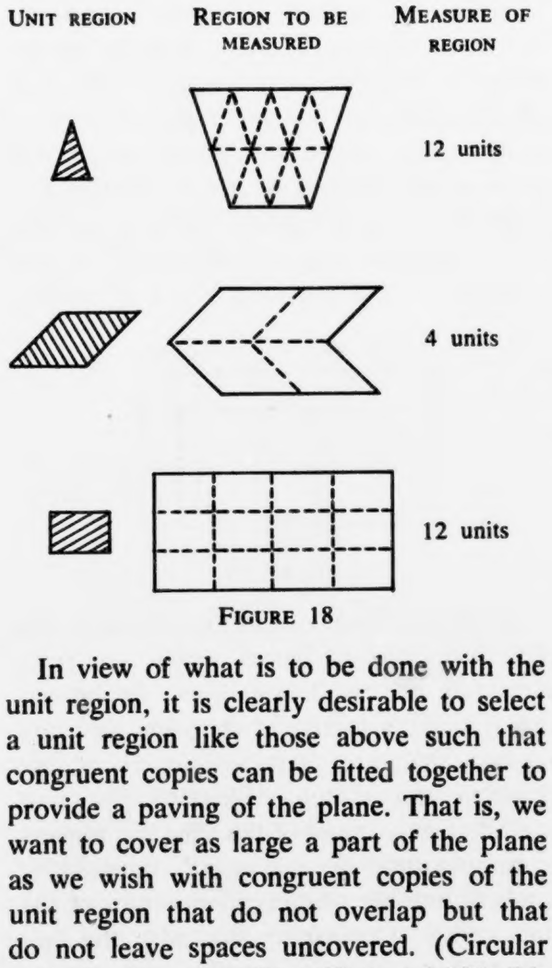


FIGURE 18

In view of what is to be done with the unit region, it is clearly desirable to select a unit region like those above such that congruent copies can be fitted together to provide a paving of the plane. That is, we want to cover as large a part of the plane as we wish with congruent copies of the unit region that do not overlap but that do not leave spaces uncovered. (Circular regions, for example, will not fulfill this purpose.)

The most common choice for a unit region is, of course, a square region each side of which is one linear unit in length; but this should not obscure the theoretical possibility of other choices and some of the interesting geometric ideas connected with paving the plane with differently shaped regions.

The measurement of a region, in the

sense just described, is called the *area* of the region. There are both similarities and differences between the measurements of area and of length. A notable similarity is the fact that measurement is still approximate. Indeed this is a characteristic of all measurement. In the examples of Figure 18, the regions to be measured were chosen so that they could be exactly covered by some integral number of congruent copies of the unit region. This, however, is clearly the exception; and in any case the process of fitting the congruent copies always involves manipulative error. Usually an attempt at covering some region with congruent copies of the unit region will show some copies inside the given region and some partly inside and partly outside. For example, using a square region as a unit, the rectangular region in Figure 19 is seen to have an area between 8 and 15 units.

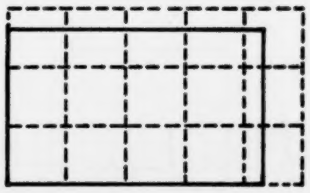


FIGURE 19

As in the case of linear measure, one may introduce measures which are fractions but not whole numbers. As before, this is really a matter of choosing new unit regions that are fractional parts of the original unit region. Whatever the unit used, however, most of the time the regions to be measured are not exactly covered by a whole number of congruent copies of the unit region. Combining this with the fact noted above that in practice one cannot construct exactly congruent copies anyway, it is clear that though in theory we may conceive a region to have an exact measure, in practice the answers we get are always approximate.

Possibly the most striking difference between the measurements of length and of area is the lack of an instrument for measuring area that corresponds to the ruler for length. Let us examine the ruler for a moment. To measure the length of a segment AB one may place the zero point of the ruler at one end, say A, of the segment and see where on the ruler the other end B of the segment falls. If, as in Figure 20, B falls at the point marked 5, this indicates at once that the length of AB is 5 units, since exactly 5 congruent copies of the unit are needed to cover AB. It is interesting to ask why this simple instrument exists, and the answer is not difficult. We have noted that measurement of length means covering by congruent copies of a unit segment, and there appears to be only one reasonable procedure for doing this. We start at an end point A of the segment to be measured and mark off a congruent copy of the unit segment in the direction toward B. If this does not yet cover the segment AB, we start at the end of this first copy of the unit and lay off another copy in the same direction. This process is continued till AB is covered. At each stage the next step is completely determined. But this is precisely the way in which the ruler is constructed. Thus, when we lay the ruler beside the segment, the marks on it indicate exactly the steps we would take if we went through the process of measuring for ourselves. The number attached to the end point of the last segment needed is then always the number of congruent copies of the unit used in covering segment AB; i.e., it is the measure of the length of AB. Thus a ruler is merely a set of congruent copies of some unit segment laid end to end along a line, each end point being associated with a number. In using the ruler, we can simply look at the number associated with the last point used, rather than having to go back and count the segments each time.

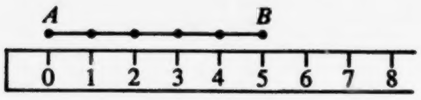


FIGURE 20

The discussion above is quite simple, and we can surely realize the saving of work in being able to read off lengths from the numbers indicated on the ruler instead of counting the number of congruent copies of the unit segment. Surely it would be equally useful to have a similar instrument for measuring areas. What is to prevent us from making one? For the ruler we covered as large a section of a line as desired with nonoverlapping congruent copies of the unit segment. In the case of the plane, we can certainly cover as large a part of the plane as we wish with nonoverlapping congruent copies of the unit region. For example, using a square region as a unit region, we can readily form a network, as shown in Figure 21.

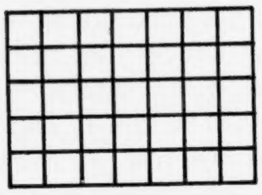


FIGURE 21

If we imagine Figure 21 drawn on a sheet of transparent plastic, we shall have a device that could (at least in theory) be fitted on any region to be measured. To find the area in terms of the given unit region, it will then only be necessary to count the number of unit square regions needed to cover the region being measured. This is indeed correct and on occasion can be very useful. In the case of the ruler, however, we were able to avoid the tedious counting process by associating numbers with the points of the scale so that it was necessary only to look for the number associated with the end point of the last segment used. Can something similar be done with our network of square regions? Unfortunately the answer is no, and for a very interesting reason. Essentially it is because of the variety of shapes of regions to be measured. Differently shaped regions may require a different order for the placement of the covering unit-square regions. Thus the order in which we would place the unit-square regions to cover the rectangular region in Figure 22A is quite different from the order needed for the region in Figure 22B. Hence if we number the square regions on our transparent sheet of plastic so that the regions numbered 1, 2, 3, 4 will fit on Figure 22A, then no placement of the plastic will have these same square regions covering the region of Figure 22B. Thus it is not possible to assign numbers to the squares on the plastic so that with proper placement an area can be read merely by noting the largest number associated with a square in the covering.

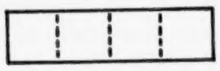


FIGURE 22A

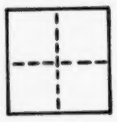


FIGURE 22B

It is this lack of a two-dimensional ruler that makes the choice of the unit square region so desirable, since, in this way, one can deduce the area of a rectangular region in the usual way by using the linear measures of the sides. Thus in Figure 23 if the length of the rectangular region is 3 units and the width is 2 units, then the area of the region, i.e., the number of unit-square regions needed to cover it, is 2 • 3, or 6.

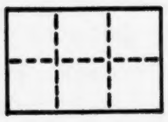


FIGURE 23

Paving the plane

As was remarked above, there are many possible choices of region that allow us to pave the plane with congruent copies. While the paving with square regions is the choice commonly made in discussing area, there is interest in considering such pavings in their own right. Pupils may find it interesting to discover as many such pavings as possible. It may be of interest to observe that a paving can be given using a unit region bounded by any triangle or any quadrilateral and some hexagons.

In addition to their inherent interest, some of the pavings suggest geometric facts that pupils will eventually see as geometric theorems. Two examples of such results will be given as a conclusion to this discussion.

Consider a paving with triangular regions. One such paving is shown in Figure 24. The triangular regions are all congruent to the unit region shown. The small numerals have been inserted in the drawing of the unit region to identify the three angles and have also been shown in part of the paving. An examination of the paving about a point as shown in Figure 24 indicates two angles congruent to angle 1, two congruent to angle 2, and two congruent to angle 3. This appears to suggest that congruent copies of angles 1, 2, and 3 (with their interiors) can be put together to fill up the space on one side of a line. Since angles 1, 2, and 3 are the angles of a triangle, this is the geometric fact that the pupil will eventually see embodied in the theorem that the sum of the degree measures of the angles of a triangle is 180.

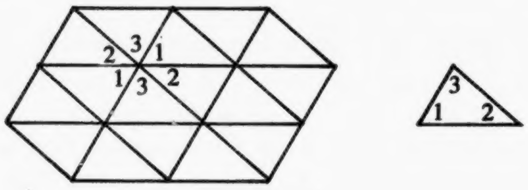


FIGURE 24

As a second example, consider the paving shown in Figure 25, with regions whose boundaries are isosceles right triangles. This is actually the familiar paving with square regions, except that each square tile has been cut by a diagonal.

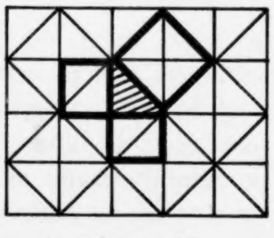


FIGURE 25

One of the triangular tiles in Figure 25 has been shaded for convenience. For each of the two shorter sides of this triangular region note the square region, which has been outlined with a heavier line. Each of these square regions consists of two of the triangular tiles. Hence if the triangular region is taken as the unit of area, each of these squares has an area of two units. Next observe the heavily drawn square whose side is the hypotenuse of the triangle. This is seen to consist of four of the triangular tiles and so has an area of 4 units. Thus the square region on the hypotenuse has an area that is the sum of the areas of the square regions on the two shorter sides. This is a special case of the famous theorem of Pythagoras that claims that this relation holds for *all* right triangles. A legend suggests that it was by looking at such a pattern that this theorem occurred to Pythagoras.

Conclusion

The major purpose of this presentation has been to exploit the idea that the concept of congruence is one of the most fundamental and fruitful ideas that arise in our intuitive perceptions of geometric relationships. In particular, it seems to provide the appropriate means for describing at the elementary school level the essential meanings for the measurement of segments, angles, and plane regions. The same ideas can readily be extended to the discussion of the volume measures of solid regions.

3

Instruction—Rationale

Techniques and methods are of doubtful worth without good reasons for their use. An effective teacher's rationale is more important than a bag of tricks. Many techniques are found in this book of readings, and nearly all of the articles have direct suggestions for teaching approaches or special methods. The articles placed in this last grouping are no exception and contain many practical suggestions for the teacher. However, these articles particularly develop direct or implied reasons for using techniques. Most of the authors in this section adhere to a similar philosophy, but each has a somewhat different point of view to express.

The initial article of this section contains a comprehensive outline of suggested content for the elementary school geometry program. Egsgard, in this paper, develops not only the scope of a program but also illustrative methods and techniques. Clearly written and concise, it presents a very fine overview of a well-planned geometry course of study.

Concern for the teacher is the focus of Inskeep's article. A brief résumé of reasons why geometry should be taught is given, followed by suggestions for the teacher who would implement geometry in his teaching. This discussion is directed to primary teachers, but it will be of value to any teacher who has not taught geometry before.

Vigilante's discussion helps to present some of the reasons, both psychological and pedagogical, why geometry should be taught. This discussion, also, is geared to the primary grades but develops a rationale that can be accepted for all grades. His essay will appeal to the reader for whom cursory mention of the need to teach geometry is not enough.

Another aspect of mathematics instruction is the part geometry plays in contributing to other areas. Robinson presents the worth of geometry and geometric approaches to other topics in mathematics. For many mathematics educators this article will furnish a most compelling reason for teaching geometry.

Skypek has investigated the thinking of Piaget. Using some of Piaget's conclusions and ideas related to the development of geometric concepts in children, she emphasizes the impact upon the curriculum. There are good reasons from a psychological point of view to introduce and develop the study of geometry early in the grades. Skypek also deals with the content that would be appropriate for the curriculum, in keeping with the theories of Piaget.

The concluding article of this section is a nicely blended combination of ideas for teaching and reasons for introducing geometry early in the experience of the child. Brune covers the scope of the curriculum, concurrently showing illustrative techniques. Moreover, he gives a logical basis for geometry in the early grades, noting the importance of informal geometric readiness to the formal axiomatic study to be undertaken in the secondary school.

Taken compositely, the articles in this section deal directly with the questions of *why* geometry should be taught in the elementary school and *what* geometry should be taught there. The rationale for the *how* of teaching is also included. The teacher will get a clear understanding of ways to involve his children in geometry. Involvement and instruction! Instruction geared to involvement! These are the objectives for which this book of readings has been compiled.

Geometry all around us—K-12

JOHN C. EGSGARD, C.S.B.

John Egsgard teaches at St. Michael's College School, Toronto, Ontario, Canada. He served as a member of the Board of Directors of NCTM, 1965–68.

There is growing evidence among mathematics educators that geometry should be experienced in each year of schooling from kindergarten through grade 12. Geometry is the study of spatial relationships of all kinds, relationships that can be found in the 3-dimensional space we live in and on any 2-dimensional surface in this 3-dimensional space. These relationships can be discovered all around us. Observe the many different shapes in your environment. This is geometry. Listen to the description of the path of the latest space rocket. This is geometry. Compare the photograph taken with a polaroid camera to the object that it pictures. This is geometry. Notice the symmetry to be found in a spherical or cubical shape and the lack of symmetry in some modern works of sculpture. This is geometry. All of these involve spatial relationships. Children are aware of spatial relationships from their earliest days. Introducing them to the idea of geometry as being concerned with shape and size in the material world will help them to realize and appreciate that mathematics is something that plays an important role in the world in which we live.

The geometry of the K–6 level should begin in 3-dimensional space with the study of solid shapes. From the earliest age the child's experience is with solids, that is, things in 3-dimensional space. In the preoperative stage, that is, K–2, the *shape* of solids should be emphasized. In kindergarten the child should play with solids of different shapes such as cubes, cones, cylinders, spheres, rectangular boxes, prisms, and pyramids. Perhaps the names of these should not be given in kindergarten, but the children should be able to sort out solids that have a similar shape when the solids have been mixed together. In order to emphasize the idea of shape, similar solids of different sizes should be used, for example, cubes with edges of 1 inch, 2 inches, 3 inches, etc. (It is interesting to note that some children do not recognize a flat cylinder, such as a round candy box, as a cylinder.) The first operation that the children should be able to perform in geometry is sorting according to shape. They should also be able to recognize things that have these shapes. For example: rubber balls are like spheres; tents like pyramids; cans and some pencils like cylinders, etc. Later they will discover how to order solids of the same and different shapes according to size and learn to measure their volume. Observe that the examination of faces of such solids will lead the children from 3-dimensional shapes to 2-dimensional shapes. The concepts of line segment and point will eventually grow from the experiences the children have with the edges and vertices of solids.

After the children have become familiar with different shapes they may use these shapes to build walls. Assignments should be given through which they may discover which shapes fit together best without leaving a gap. In this way they will find some of the properties of these shapes. Here are some examples of assignments. (Some of the assignments are similar to those found in the book "Shape and Size" of the Nuffield Project.) In doing these assignments, four or five children may work together. Each group will have 24 of each shape of a given size, namely, cubes, rectangular boxes, spheres, prisms, cylinders, pyramids, etc.

Using all the bricks of one kind, try to build a wall that has two thicknesses of brick.
Repeat using all the different shapes.
What shapes are most easily used for building walls?
Can you say why this is so?

In some groups all of the children will decide to work on the same wall. In other groups different children will work on different walls at the same time. In any case, the children will find that the cube and rectangular-shaped brick fit together best. These assignments will help them recognize that this happens because these bricks can occupy the same space in several different ways.

Use the cubes to build a wall 2 bricks thick.
Take out a brick from the wall, turn it around, and replace it in its hole.
How many different ways can you find to replace a brick?
Marking the faces of the brick in different ways will help.

Use the rectangular bricks to build a wall 2 bricks thick.
Take our a brick from the wall, turn it around, and replace it in its hole.
How many different ways can you find to replace a brick?
Are rectangular bricks or bricks of cubes ordinarily used to build walls?
Examine a wall to see how these bricks are fitted together.
Why do you think this type of brick gives a stronger wall?

Observe that the turning around of the bricks is the beginning of the study of symmetry under rotation. It also has a more immediate purpose as the following assignment indicates.

Examine the wall you have built. Look at the corners of the bricks.
What type of corners do the cubes and rectangular bricks have?
Why are these shapes used in building walls?

The ideas brought out will probably include the facts that the bricks have square corners that fit together well, that the bricks can be replaced easily, that it is easy to make the top of the wall level, etc. These ideas can be investigated further in the classroom. The teacher will have to introduce the children to the term "right angle" for the "square corner" of rectangular faces and show them how to make a "right-angle tester" by folding a piece of paper twice (fig. 1).

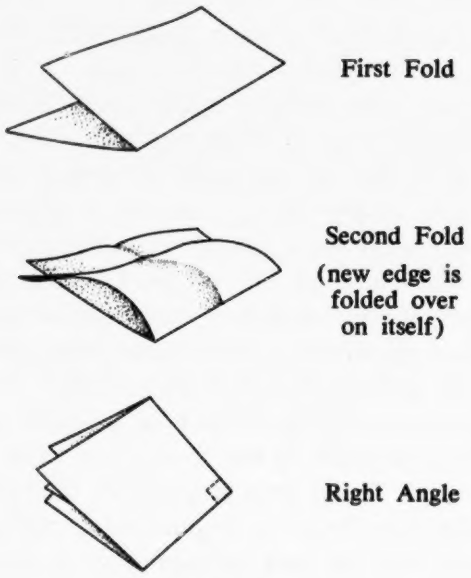


FIGURE 1

The children can be asked to try to find other right angles in the classroom and about the school and to test them by fitting their paper right angle onto the shapes.

Through these and similar assignments the children learn how to distinguish the different 3-dimensional shapes and discover some of the simpler properties of the shapes. After these shapes have become familiar, the children are ready to consider size and measure of volume. Children can be led to the notion of volume through the process of sorting similar shapes by size. Comparison of size can be made by filling hollow shapes with things such as water, peas, beads, pebbles, cubes, and sand. Assignments similar to the following can be made.

You will need a large jar and a paper cup.
Guess how many cupfuls of water you will need to fill the jar.
Pour water into the jar a cupful at a time until it is full.
How many cupfuls of water did you use?
How close was your guess? Did you guess too many or too few?

Other assignments can be given using a shoe box, a small match box, some sand, and also marbles and a glass jar. Students will realize that marbles are not very good for finding out how much space there is in a jar. Ultimately the children will come to understand that two cylindrical jars, for example, are the same size (that is, have the same volume) if the same number of spoonfuls of sand are needed to fill each, and that a jar that needs a greater number of spoonfuls of sand to fill it is larger than either of these. Once they can do this they can sort any given set of hollow shapes according to size. Note that the sorting of sizes in this way leads to the notion of units for measuring volume. The unit in the last case is a spoonful of sand. The standard units come at a much later stage, as does the formula $\ell \times w \times h$.

So far we have seen that the study of 3-dimensional space begins with the sorting of solid shapes, is followed by the ordering of solids by size, which leads to the measurement of volume. A similar progression should be followed in the study of 2-dimensional space as well: shape to size to measure. Nevertheless the examination of 2-dimensional shapes should begin immediately after the children have become familiar with different 3-dimensional shapes and before they are able to sort solids according to size. In the handling of solids they will discover that some solids have flat surfaces, some have curved surfaces, and some have both flat and curved surfaces. When they encounter the cylinder as they sort solids according to the type of their surfaces, flat or curved, they will gain one of their first introductions to the idea of the intersection of sets. The first 2-dimensional shapes that they should be able to recognize are found as faces of the 3-dimensional solids they have been using—namely, the square, the rectangle, the triangle, the circle. An assignment such as the following will help the children to recognize and sort different kinds of 2-dimensional shapes. An inked stamp pad, a set of solid shapes, paper, and scissors will be needed.

Take one of your solid shapes that has a flat surface.
Press one of its flat surfaces on the stamp pad.
Print a picture of this shape on a sheet of paper by pressing the inked face on the paper.
Do the same for the other flat surfaces of your solid shape.
Cut out the different pictures of faces from the paper.
How many pictures do you have?
Are any of these pictures of the same shape? Which?
Repeat with a different solid shape until all are done in this way.
Write about your results.

Eventually the children will come to relate a specific set of 2-dimensional shapes with each 3-dimensional shape and be able to do the following assignment (fig. 2).

Use your set of solids to decide which of the three solids in the top space has been used to trace the set of faces in the bottom space.

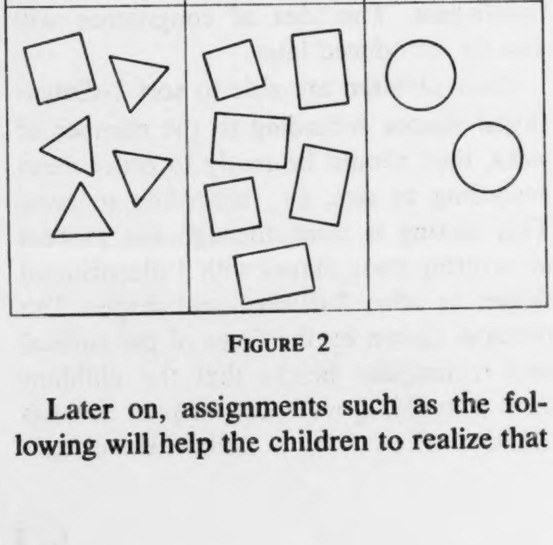


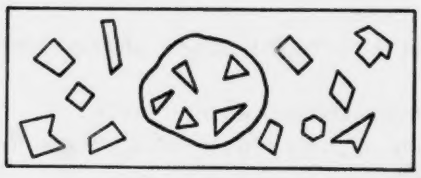
FIGURE 2

Later on, assignments such as the following will help the children to realize that

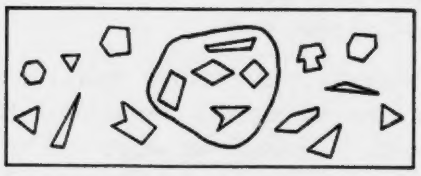
2-dimensional shapes can be classified according to the number of sides. They should be given a collection of polygons made from colored cardboard with three sides, four sides, five sides, etc.

Take the set of shapes and sort them into subsets like this: those with 3 sides, those with 4 sides, those with 5 sides, and so on. Make a loop of string around each of your subsets, or draw chalk rings around them on the floor.
What is the name of the subset of shapes with 3 sides?
Do you know the name given to the subset of shapes with 4 sides?

The children will do something like this.



Subset of Triangles



Subset of Quadrilaterals

In this way the children can learn the names such as triangle, quadrilateral, pentagon, hexagon, etc. Names such as equilateral triangle, regular pentagon, parallelogram, and rhombus come at a later stage after the children have learned something about the measure of length and parallelism. The idea of congruence will also be introduced later.

Once children are able to sort 2-dimensional shapes according to the number of sides, they should be ready to order them according to size, i.e., according to area. This sorting is done through the process of covering these shapes with 3-dimensional shapes or other 2-dimensional shapes. The patterns shown by the faces of the cubical and rectangular bricks that the children used in building walls can help to develop the idea of covering a 2-dimensional surface. Most of the tiles that children see in walls, the floors they walk on, the sidewalks they jump on to and from school use rectangular shapes, so the covering work in the early stages will be largely concerned with these shapes. When children are looking at a wall to study the brick pattern, they can be asked such questions as "What shape do you see?" (Rectangles.) "How could you make a pattern like this using bricks?" Some answers that will be received will be: "Draw around faces of bricks"; "Use squared paper"; "Cut out shapes from colored paper and arrange them in patterns," and so on. Groups of children can be given square and rectangular tiles made from colored cardboard. The following assignment can be made.

Make tile patterns by fitting the square shapes together.
How many patterns can you make using the rectangles?
How many patterns can you make using the squares and rectangles together?

Once the children have the feeling for covering surfaces, they can begin to compare the size of surfaces using assignments like the one below. Shapes such as squares, equilateral triangles, regular pentagons, regular hexagons, circles should be available with a sufficient number of each of the same size to be able to cover the surfaces being used.

Take all of the squares. Guess how many you will have to use to cover the front of your workbook.
Now use the squares to cover your book.
How many did you use? Was your guess too large, or too small?
Repeat with other shapes. Use only triangles, circles, hexagons, rectangles, and so on.
With which of these shapes did you find you could cover the surface?
Which shapes were not very good for covering the surface?

The children are then given several books or similar 2-dimensional shapes having different areas of surface.

Order these books by finding the size of the surface of the front cover.
Which cover is largest? Which is smallest?
Write about this in your own way.

These assignments should culminate in a discussion of the various ways the shapes are used to cover the surfaces. This discussion should lead to the idea of using squares, triangles, regular hexagons for covering surfaces. The reason why pentagons and circles are not useful should also arise. By analogy with the use of the cube for 3-dimensional space, the children will see that the square is probably best. Later the area of a surface will be measured by the number of standard squares required to cover the surface, so that assignments can be given in which the children use paper marked in squares, say one inch or one-half inch.

Cover the front of your book with the square paper and find how big it is by counting squares. Now do the same on the cover of a different book. Which cover was bigger? How do you know?

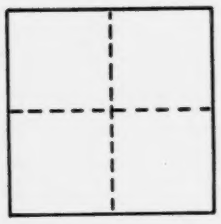
Draw around your right hand on the squared paper. When your group has done this, find out whose hand covers up the biggest surface on the paper.

Similar assignments can be given where the children measure the surface area of a leaf, a cylindrical can, a box, and so on. Once again, it is important to point out that no formulas have been used to determine area. The formula $\ell \times w$ for a rectangle will be discovered later.

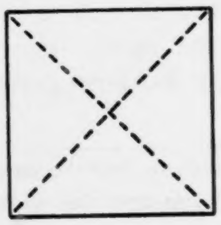
So far I have concentrated on that part of geometry for K–6 that can be called the study of shape, size, and measure. Occasionally I have made reference to symmetry and the transformation of rotation. I shall now look further into the study of transformation in grades K–6. The tile patterns used in the study of area are important aids to the understanding of transformations. When studying the making of tile patterns and the covering of surfaces, assignments such as the following can be added to introduce the notion of symmetry.

Trace around the square shape. Trace around the other rectangular shape.
Cut out the shapes from the paper.
How many ways can you fold these paper shapes so that one half matches the other half?

The children should be asked to describe what the shape on each side of the fold looks like. Most will find this way of folding:



Some will find this way of folding:



The same type of work can be done with other 2-dimensional shapes such as triangles and pentagons, but at a later stage in the development. The idea of symmetry or balance in shapes should be investigated in other ways. One avenue for exploration is pattern work in arts and crafts (fig. 3). Another is in the study of blot patterns (fig. 4). In any case the children should try to find the line or axis of symmetry.

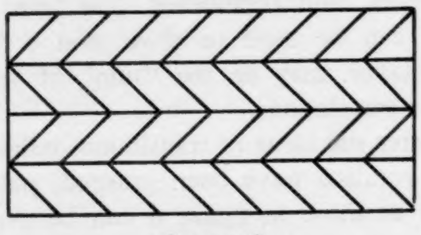


FIGURE 3

Observe that a line of symmetry is also a mirror line for a reflection transformation.

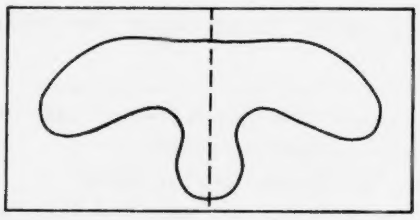


FIGURE 4

The following assignments help introduce the notion of translation. The children are given a cardboard square and these three patterns in turn (fig. 5). When a figure is moved along a straight line without turning the figure, then a translation transformation has been performed on the figure.

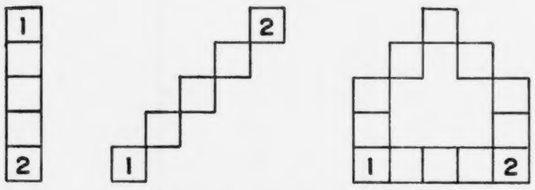


FIGURE 5

How could you show that the cardboard square is the same shape and size as each square in the pattern?
How could you use the cardboard square to trace out a pattern like the given one? Trace out the pattern.
How many different paths can you find along which you can slide the cardboard square from position 1 to position 2, so that the cardboard is always passing over a square in the pattern? You must not lift the square from the page nor turn it around.

Similar questions can be asked for patterns of the same kind that contain rectangles, triangles, and rhombuses. The third pattern can be used to show that a single translation may be the "sum" of several other translations.

After the ideas of translation, reflection, and rotation have been grasped, patterns such as those in figure 6 can be used to emphasize the differences among the three. For each of the patterns I, II, and III, the children will be given a piece of colored cardboard of the same shape and size of each shape in the pattern and will be asked these questions:

Is your cardboard shape the same shape and size as any shape in the pattern?
Which? How do you know?
What type of transformation will take your cardboard shape from position 1 to position 2? From position 1 to position 3? From position 1 to position 4?

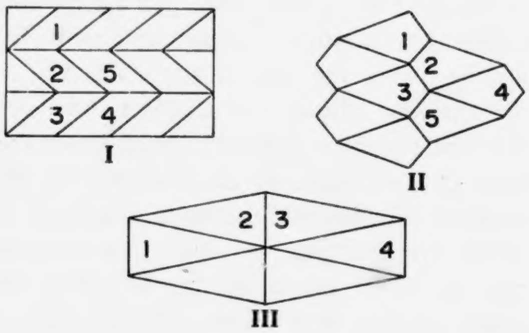


FIGURE 6

In pattern I, a reflection in the line common to positions 1 and 2 carries the shape from 1 to 2, while a translation carries it from 1 into 3 or 4. In pattern II, a rotation about the midpoint of the segment common to 1 and 2 brings the shape from position 1 to 2. A translation will take the shape from 1 to 3 or 4. In pattern III, a rotation about the midpoint of the segment common to 1 and 2 will carry the shape from 1 to 2. A translation carries it from 1 to 3 and a rotation or reflection from 1 to 4.

The idea of composition of transformations can be introduced with these patterns by asking for the succession of transformations needed to go from position 1 to position 5 in patterns I and II.

In pattern I a reflection and a translation are most obvious; in pattern II a rotation and a translation or a reflection are sufficient; in pattern III a rotation, a translation, and a reflection are usually selected. The idea of composition of transformations can be introduced by asking for the transformation or transformations needed to go from position 1 to position 5 in patterns I and II. In pattern I a reflection to position

2 followed by a translation to position 5 is usually suggested first. In pattern II a rotation from position 1 to position 2 followed by a translation to position 5 will be selected by most children. In pattern II some children will see that it is possible to rotate from position 1 to position 5 using only one transformation.

Assignments such as the preceding, together with discussion should be means to the clear understanding in grades K–6 of the concept of transformation. This knowledge will prove of considerable use in the grade 7–10 level. For example, assignments can be made that will lead the children to discover certain properties of the parallelogram and the rhombus. Strips of cardboard with a hole punched about a quarter of an inch from each end will be needed. There should be at least four strips of two different lengths for each child.

Fasten four strips of the same length, using paper fasteners, to make a rhombus.
Place this on a sheet of paper. Trace around the inside of the framework. Cut out this rhombus shape.
How many axes of symmetry has this shape?
Fold it along one of the axes of symmetry so that one half matches the other.
What can you discover about the lengths of the sides that fit on to each other?
What can you discover about the angles?
Now fold it along the other axis of symmetry.
What does this tell you about the sides and angles this time?

The children should discover that the angles at the opposite corners are congruent. Some may even notice that the diagonals meet at right angles.

Rotation can be used to help the children discover a similar property for the parallelogram.

Fasten four strips together as before, two of one length and two of another length, with strips of the same length opposite to each other to form a parallelogram.
Use the strips to make different parallelogram frameworks.
Draw some of these shapes on thin cardboard by drawing along the inside of the strips.
Cut out the parallelograms drawn on the cardboard.
Now place the cutout shape on a sheet of unlined paper and draw a frame around it.
Discover how many ways you can fit the cardboard shape into its frame without turning the shape over.

Discussion should bring out that "half a complete turn" or a complete turn will do this. Immediately following the discussion the following assignment should be completed.

Fit one of the cardboard shapes into its frame.
Now make a half turn with the shape so that it fits into the other half.
Look at the angle colored red. Does it fit into its new position?
Does the angle colored blue fit into its new position?
Repeat for several of your cardboard shapes.
What can you discover about the angles of a parallelogram from this?

Children at the grades 7 and 8 level can also use the composition of transformations to get their first introduction to the group properties without making use of a number system. For rotations of a square about its center, closure is exemplified by the fact that a rotation of 90° followed by a rotation of 180° is equivalent to a rotation of 270° (fig. 7).

The lack of the commutative property

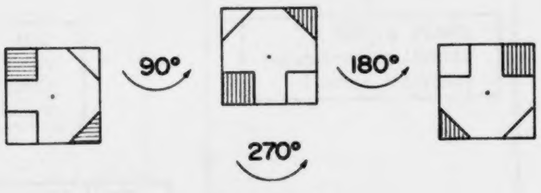


FIGURE 7

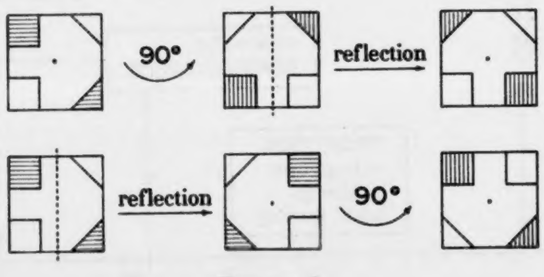


FIGURE 8

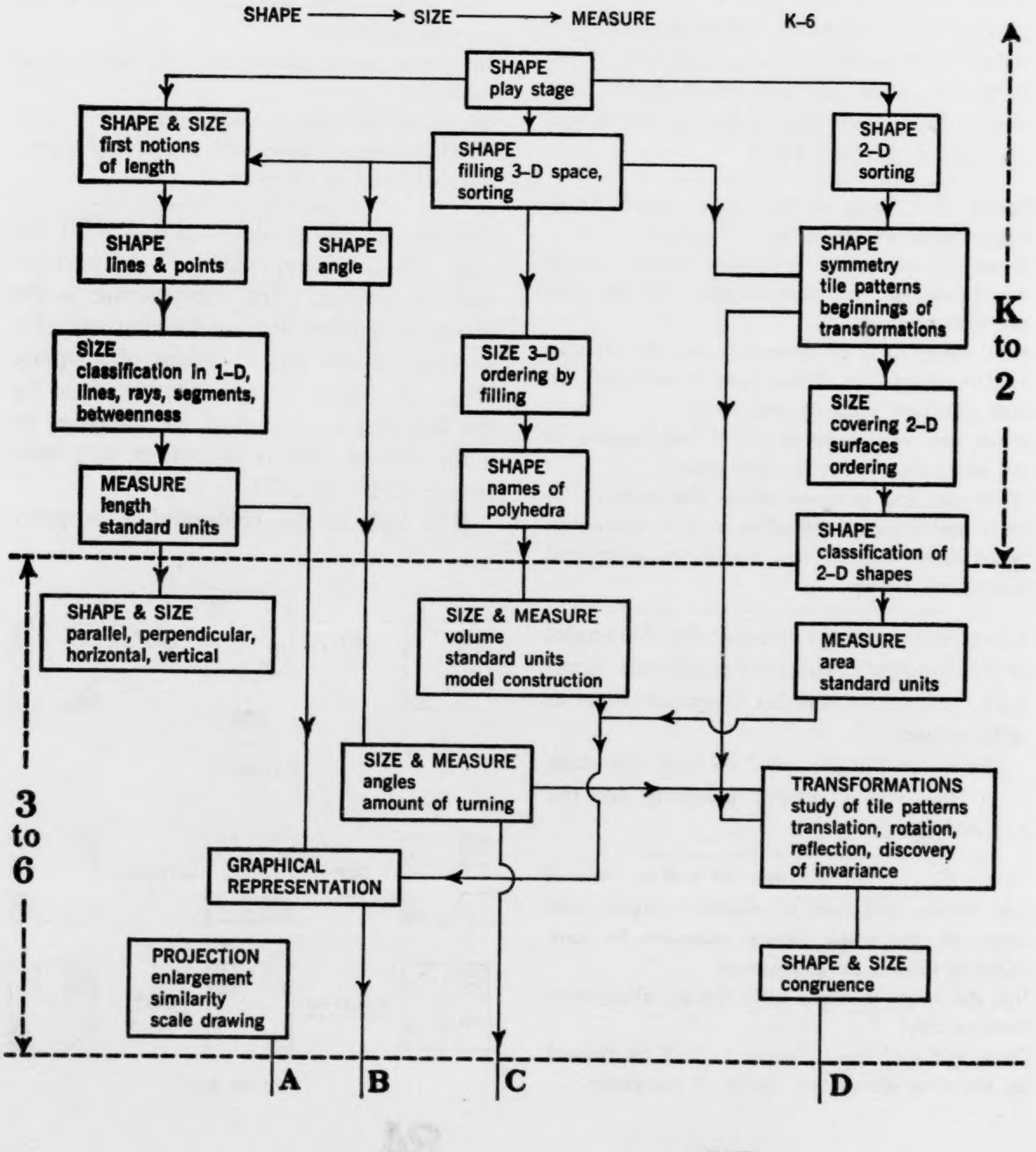
under composition is evident from the fact that a rotation of 90° followed by a reflection about a vertical axis of symmetry does not give the same result as a reflection about a vertical axis of symmetry followed by a rotation of 90° (fig. 8).

Children have great fun playing with square cards to test these and the other group properties.

Let us return to grades K–6. The following chart summarizes what I think should be taught in geometry at this level and indicates some of the relationships among topics (chart 1).

The next chart is a continuation of the first and is concerned with grades 7–10 and 11–12. Vectors can be introduced as soon as the graph of a point is understood. The section on deductive proof in grades 7–10 is not to be considered as a formal organization of Euclidean geometry. Rather there should be short sequences of related theorems based on the congruence and parallel facts established in earlier grades.

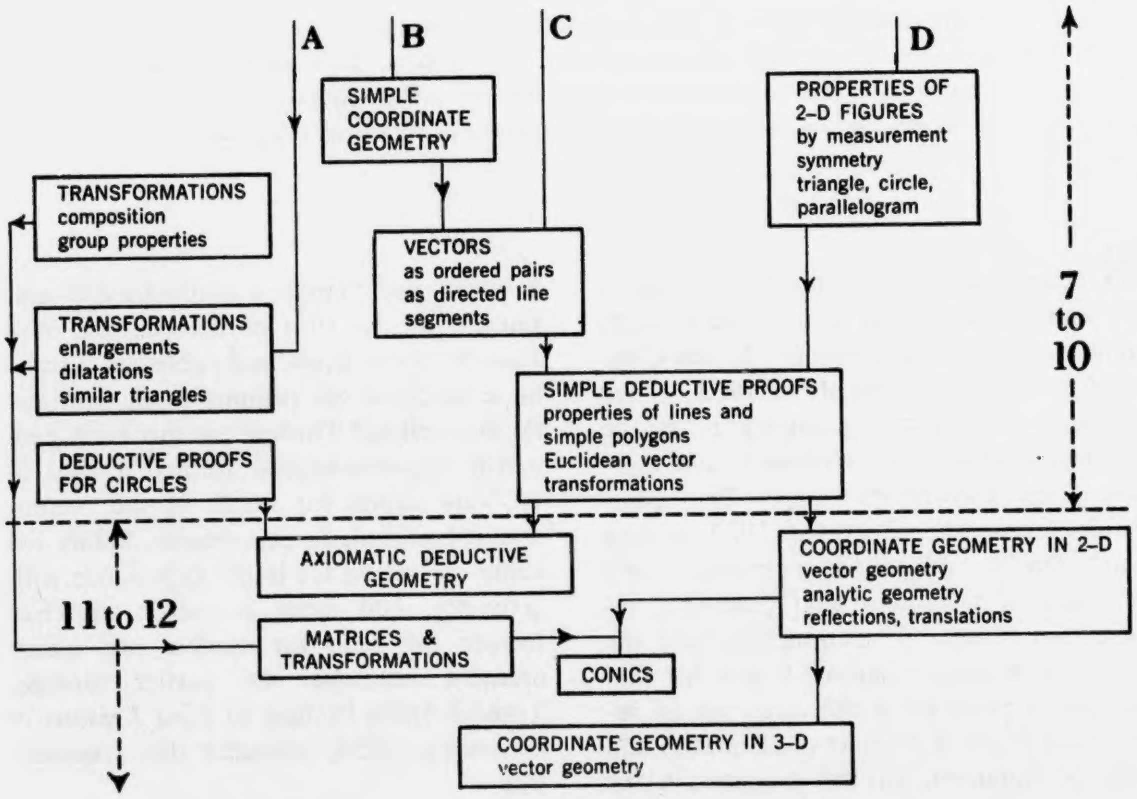
CHART 1


SHAPE → SIZE → MEASURE
K–6
SHAPE play stage
SHAPE & SIZE first notions of length
SHAPE filling 3-D space, sorting
SHAPE 2-D sorting
SHAPE lines & points
SHAPE angle
SHAPE symmetry tile patterns beginnings of transformations
SIZE classification in 1-D, lines, rays, segments, betweenness
SIZE 3-D ordering by filling
SIZE covering 2-D surfaces ordering
MEASURE length standard units
SHAPE names of polyhedra
SHAPE classification of 2-D shapes
K to 2
SHAPE & SIZE parallel, perpendicular, horizontal, vertical
SIZE & MEASURE volume standard units model construction
MEASURE area standard units
SIZE & MEASURE angles amount of turning
TRANSFORMATIONS study of tile patterns translation, rotation, reflection, discovery of invariance
3 to 6
GRAPHICAL REPRESENTATION
PROJECTION enlargement similarity scale drawing
SHAPE & SIZE congruence
A
B
C
D

Axiomatic deductive geometry has been delayed until grade 11 or 12 and this for the best students. I am inclined to agree with the British view that it is suitable for the top 5 percent or less of the student body.

The learner can be challenged to interpret the tangible world of spatial relationships that exist in his environment. Discovering these relationships will help learners interpret and appreciate mathematics. The simple ideas can lead to the abstraction of geometrical ideas of space and size.

CHART 2


A
B
C
D
SIMPLE COORDINATE GEOMETRY
PROPERTIES OF 2–D FIGURES by measurement symmetry triangle, circle, parallelogram
TRANSFORMATIONS composition group properties
VECTORS as ordered pairs as directed line segments
7 to 10
TRANSFORMATIONS enlargements dilatations similar triangles
SIMPLE DEDUCTIVE PROOFS properties of lines and simple polygons Euclidean vector transformations
DEDUCTIVE PROOFS FOR CIRCLES
AXIOMATIC DEDUCTIVE GEOMETRY
COORDINATE GEOMETRY IN 2–D vector geometry analytic geometry reflections, translations
11 to 12
MATRICES & TRANSFORMATIONS
CONICS
COORDINATE GEOMETRY IN 3–D vector geometry

Primary-grade instruction in geometry

JAMES E. INSKEEP, JR.

San Diego State College, San Diego, California

Dr. Inskeep is a professor at San Diego State College and is the associate editor of THE ARITHMETIC TEACHER. *He is well known to our readers through his editorial comment, "As We Read." This article grew out of a paper that he presented in November 1967 at the National Council of Teachers of Mathematics meeting at Richmond, Virginia.*

Recommendations for teaching geometry to young children run as a thread through much of the recent literature. In the Cambridge Conference Report of 1963, it was recommended that "geometry is to be studied together with arithmetic and algebra from kindergarten on."[1] The Cambridge Conference Report of 1967, dealing with teacher education, suggested early merging of arithmetic and geometry, instruction in spatial relationships, and the study of standard shapes.[2] It was felt that geometry provided a rich area for an integrated study of abstract mathematics and the environment. In the preparation for statewide adoption of mathematics texts, the State of California recently reaffirmed the need for geometry as an integral part of the primary-grade curriculum.[3] Other states and groups have made similar recommendations.

It might be inferred that this desire for primary-grade geometry is confined to postwar and "modern mathematics" emphases. At the turn of the century, William W. Speer wrote and published a small book designed for primary-grade arithmetic instruction.[4] Throughout this book geometric experimentation formed a kind of unifying strand for much of the recommended arithmetic experiences. Solids became the media for initial experience with geometry, and other geometric exercises formed the basis for number and measurement concepts. Of earlier vintage, Thomas Hill's Preface to *First Lessons in Geometry,* 1854, included this commentary:

> I have long been seeking a Geometry for beginners, suited to my taste, and to my convictions of what is a proper foundation for scientific education. . . . Two children, one of five, the other of seven and a half, were before my mind's eyes all the time of my writing; and *it will be found that children of this age are quicker of comprehending first lessons in Geometry than those of fifteen.*[5]

There is considerable agreement as to the desirability of including geometry in the primary grades.

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[1] Educational Services Incorporated, *Goals for School Mathematics: The Report of The Cambridge Conference on School Mathematics* (Boston: Houghton Mifflin Co., 1963), p. 33.

[2] Educational Development Corporation, *Goals for the Mathematical Education of Elementary School Teachers: The Cambridge Conference* (Boston: Houghton Mifflin Co., 1967), pp. 8–9, 100–101.

[3] California State Department of Education, *Mathematics Program, K-8: 1967-68 Strands Report, Part 1* (Sacramento, Calif.: The Statewide Mathematics Advisory Committee, 1967). Also *The Bulletin of the California Mathematics Council,* XXV (October 1967), 5–15.

[4] William W. Speer, *Primary Arithmetic: First Year. For the Use of Teachers* (Boston: Ginn & Co., 1897), as reproduced in 1940.

[5] Thomas Hill, *First Lessons in Geometry* (1854), Preface, as quoted in *The Teaching of Geometry,* Fifth Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: The Council, 1930), p. 10.

Evidence in support of these recommendations for geometry in the primary grades may be categorized into three classes: (1) primary-grade children's ability to learn geometry, (2) the existence of successful ongoing projects in primary-grade geometry, and (3) the intrinsic worth of geometry. As discussed in this paper, the primary grades are considered to be the preschool (including kindergarten) and the first three grades of the elementary school. The remainder of the discussion is organized to include the above three categories, implementation of geometry instruction in the primary grades, and teaching suggestions.

Children's ability to learn geometry

The most impressive evidence as to the ability of primary children to learn geometry comes from the work of Jean Piaget.[6] Piagetian results also indicate the form and content of these early experiences. Two generalizations from his work are worth noting. The first of these deals with the potential content of early instruction. Children understand topological ideas first, followed by those of projective geometry, and then they finally grasp Euclidean concepts. The topological concepts include the ideas of proximity, order, enclosure, and continuity. Children are able to determine the interior and exterior of such closed curves as a boy's marble ring or a girl's hopscotch pattern. Shapes and lines, including the number line and the idea of order, can be grasped by primary-grade children. Ideas of measurement associated with Euclidean geometry come later. For most children, the ideas of our formalized geometric systems will be preceded by an intuitive development of simple topological concepts singled out and fused into the less generalized projective and Euclidean ideas.

Another important finding from the work of the Geneva school is the consistent statement for developmental learning. Children will learn by manipulating their environment, and geometry can provide the vehicle for this manipulation. Intuitive ideas follow and depend upon the encounter of the child with his environment. For this reason, a primary-grade teacher will probably give children solids to touch and feel. The primary-grade experience will also include much experimentation with the visual presentation of shapes as well as the manipulation of objects.

The results of other research tend to support the fact that children can and do learn geometry at an early age. The emphasis given here to the Geneva school is due to the direction which the results give for introducing and teaching geometry to children. In addition to the work of educational psychologists, there is a growing body of data from teachers to indicate that children can handle geometry well and without harm to other emerging concepts. Some of these data will be dealt with in the section that follows.

Projects in primary geometry

The British and Canadian educators have done considerable experimentation with primary-grade geometry. There are also schools and groups in this country taking active parts in the development of geometry for the primary grades. However, only two projects will be noted in this section, both of which reflect the influence of the Geneva school of educational psychology. These two projects are (1) the English work as described in the publication of The Schools Council, *Mathematics in Primary Schools,*[7] and (2) the Ontario Geometry Project.[8]

The English suggestions cover more

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[6] For a resumé of these and other findings of Piaget, see John H. Flavell, *The Developmental Psychology of Jean Piaget* (Princeton, N.J.: D. Van Nostrand Co., 1963), pp. 327–41.

[7] The Schools Council, "Mathematics in Primary Schools," *Curriculum Bulletin No. 1* (2nd ed.; London: Her Majesty's Stationery Office, 1966).

[8] Ontario Institute for Studies in Education, *Geometry: Kindergarten to Grade Thirteen* (Toronto, Ontario: The Institute, February 1967). Also *Ontario Mathematics Gazette*, Special Elementary School Edition, September 1966, pp. 5–13, 42–48; October 1967, pp. 7, 8, 14–24, 27–33; and THE ARITHMETIC TEACHER, XIV (February 1967), 90–93, 136–40.

than geometry and give a completely modern flavor to the curricular offering. Some of the major topics covered in this project are shapes, dimensions, symmetry, similarity, and some work with limits through a geometric approach. Emphasis is placed upon the learning of concepts by the manipulation of the environment and by direct experience with constructive and sensory activities. Some of these intriguing activities include working with tiles and shapes to cover areas and the handling of solids with attendant description of faces, edges, and vertices. Among the older primary-age groups, some intuitive grasp of limits is developed by reference to the spiral and sets of contracting (or expanding) squares.

The Canadian experience was somewhat different in that specific attention was devoted to geometry as such. The strand of geometry was drawn from kindergarten through high school. The geometry developed for the primary grades included class instruction in single topics, group work with "assignment cards," and flexible grouping for exposure to many experiences. The emphasis was placed upon individual experience with mathematical models and personal experimentation by the children. Classification of shapes by size and other attributes, work with geoboards, and applications to art were among the activities in which primary children participated. An interesting comment was made by one of the Canadian primary teachers participating in the project: "Geometry became the children's favourite subject. Why? 'Because,' they said, 'we can do it ourselves.' "[9] The results of both of these projects should challenge teachers to teach geometry in the primary grades.

Intrinsic worth of geometry

There are two major intrinsic reasons for considering geometry as a primary-grade topic: (1) there is desirable mathematical content to be derived from a study of geometry early in the child's experience; and (2) children are surrounded by objects which have geometric significance.

Geometry provides a sound mathematical background for children. Many topics which are treated in the intermediate grades and secondary schools lend themselves to geometric interpretation. For many children, the only geometry they receive is that offered in a secondary school formal course. By this time the potential usefulness of initial experience in geometry has been lost. In addition to readiness, every child should have some geometry to be mathematically literate. Regardless of his plans for college or vocation, there are some geometric ideas which should be in his common knowledge. Many of these ideas can be introduced in the primary grades. Geometry is desirable mathematics for young children.

Geometry is everywhere! This may be an exaggeration, but it is evident that we live in a sea of geometric shapes. We are inundated with terms and phrases which have geometric significance. The Pentagon in Washington is an example, as is the village square. Nature also weaves a beautiful, nonabstract array of geometric models. Parallel lines are seen in architecture and in the spacing on theme paper. Modern slang includes the term "cube," and the delta wing suggests a particular shape. The list of applications in art, music, and science is nearly endless, and we live in homes or work in offices where shapes of one sort or another enclose us. Environmental geometry certainly has a place in the curriculum.

Implementation

Some readers may feel convinced that geometry should be taught but still hesitate to implement their convictions. For this reason, the following suggestions are given as helps in getting a geometry program started in the classroom. The following four recommendations are not necessarily equal in substance or possible results.

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[9] Sandra Rivington, "A Primary Teacher's Impression," *Ontario Mathematics Gazette*, October 1967, p. 7.

Experiment with a modest lesson at first. A small dosage can be a kind of pilot study for the particular experiences the teacher feels he can handle. Ideas from workshops, journals, student teachers, and other sources may give incentives. Then the steps to implement the teaching can be taken deliberately and carefully. One well-executed lesson is a stimulus for others.

Set aside a special day specifically for geometry. This suggestion is made for teachers who like to tackle one thing at a time. Geometry is probably best integrated into the total mathematics program, but experimentation in a concerted fashion may give the teacher the confidence he needs to weave it into the rest of his program.

Integrate geometry into your lesson plans. This suggestion requires more sophistication and dedication than the two previous ones. A teacher will need to plan his program in such a manner that geometry is fitted into his regular schedule. He may decide to use a special day, or he may have "quickie" lessons at intervals. The important point is that geometry should become a functional part of the entire mathematical offering.

Actively seek out information of the content and pedagogy of geometry. Without ideas and some knowledge of the subject, many teachers will hesitate to take any of the previous suggestions. For this reason, this last suggestion is crucial to the implementation of *any* geometry in the classroom. Some sources include members of the local mathematics council, teachers in other grades, and interested administrators. THE ARITHMETIC TEACHER is a good source for content as well as pedagogy.[10] Once a teacher gets a taste for the subject, he will find other sources as well. With background in content, a teacher will find the topic a challenge to his ability to teach it.

In the final section of this paper some suggestions for teaching geometry are given. The reader will wish to use his own creativity in extending their usefulness.

Teaching suggestions and techniques

The most important suggestion for teaching geometry is neither specific nor comforting. *Do some personal research* with your own children and develop your own techniques. This will seem obvious, but it will produce far more satisfying results than all the "copied" or "adapted" ideas you may gain.

A more specific suggestion for the primary-grade teacher is to *start with simple objects.* As has been pointed out, the environment is rich in geometric models. Capitalize on these. Look for shapes in the room. Compare shapes in dress patterns, room decorations, books, and other pictorial material. Ask the children to bring shapes from home such as are found in the leftover scraps of mother's new dress or the remainder ends of wallpaper. There are many examples of shapes such as the circle, square, rectangle, and triangle to be obtained by simply searching for them. Simple solid objects can also be used in a creative manner. With common building blocks such as are found in many primary classrooms, faces and "points" may be counted or described. It is probably wise to begin with solids for plane-figure identification. This type of instruction differs from the formal approach to geometry via the point-, line-, shape-definition route. Give children manipulative experience with shapes and solids, and the definitions will come quite naturally. Proper mathematical names play an important part in your instruction as the need arises to talk about the children's discoveries.

Organize the classroom for small-group experimentation. Try arranging your children in small groups as was done by

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[10] A copy of an annotated bibliography of recently published articles of THE ARITHMETIC TEACHER that pertain to the teaching of primary-grade geometry may be obtained by request from the author, San Diego State College, San Diego, California 92115.

Black.[11] (There were forty-two children in her classroom!) This suggestion will require extra effort at first, but once some of the tasks and assignments are developed, the work will become less burdensome. The children in Black's room were sent to small-group centers, each of which had a separate activity. Some centers included sets of solids to examine and describe. Others had game-like activities and graphing. For each of the activities there was adequate opportunity to experiment and discover ideas.

Another idea, which is hardly novel, deals with the room environment. *Arrange your room to reflect geometry.* Use the bulletin boards for display and for the children's work. Such titles as "Watch for the Mystery Shape!" and "What's My Line?" may serve to stimulate ideas for bulletin board arrangements. Encourage the children to display and categorize shapes that have been found in magazines and newspapers. Teachers may find pictures of bridges, modern engineering feats, and interesting architecture to enhance the room environment. Do not neglect the study center! On a table to one side of the room the teacher may wish to have games and assignment cards which the children may use when other work is finished.

Procure materials which can be used to teach geometry. Some of the materials and devices which should become part of your teaching aids are as follows: geoboards of various sizes and shapes, pegboards, solids (homemade, "home-collected" or commercial), graph paper of various types and sizes, and assorted large pictures that have geometric applications. Other materials will be discovered as the teacher gives attention to the problem of teaching geometry. If available, an overhead projector can provide the teacher with another means for visual presentations.

As a final suggestion, *combine the teaching of geometry with other areas* of the curriculum. The art period certainly lends itself to the teaching of geometry. Even the youngest child can be given some tactile or sensory experience in art through shapes and solids. Social studies and science lend themselves to geometry. Representations of various numerical ideas, graphs of simple designs, maps, and indications of size occur frequently in these areas. Look for geometry. The teacher who emphasizes the aspect of "looking for something" will find his children "seeing it."

In conclusion, the following quotation from Thomas Hill's Preface expresses the philosophy of this paper.

> I have addressed the child's imagination rather than his reason, because I wish to teach him to conceive of forms. The child's powers of sensation are developed before his powers of conception, and these before his reasoning powers. . . . I have, therefore, avoided reasoning and simply given interesting geometric facts.[12]

Hill's developmental "stages" are very similar to those of certain psychologists today, even though his terminology may not be that currently in use. His appeal to the intuitive experience, his approach to informal exercises, and his willingness to permit children to experiment with geometry are evident. However, no teacher will work with primary-grade geometry unless he knows the subject himself and is made aware of the possibility of teaching it. One article can only raise interest. The test of interest is the teaching of some geometry in the classroom. Enter into this most exciting experience. Teaching geometry is fun!

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[11] Janet M. Black, "Geometry Alive in Primary Classrooms," THE ARITHMETIC TEACHER, XIV (February 1967), 90–93.

[12] Hill, *op. cit.*

Why circumvent geometry in the primary grades?

NICHOLAS J. VIGILANTE
University of Florida, Gainesville, Florida

Professor Vigilante has been teaching functional arithmetic courses at Otterbein College, Ohio. His new duties at the University of Florida include teaching elementary mathematics courses and supervising student teachers in the internship program.

Today the arithmetic program in the elementary grades is being reevaluated. New ideas and approaches are being proposed, discussed, and tested. One of these ideas, the teaching of geometry in the elementary school, is being introduced as early as kindergarten. One of the most extensive and well-known programs in geometry for the primary grades has been developed by Patrick Suppes and Newton Hawley at Stanford University. Their program is more formalized than the experiences some children have in their present classrooms. The presentation of geometry differs from one classroom or school system to another. Some teachers find it very essential; others are still questioning its value.

Is teaching geometry in the primary grades really worthwhile? There is much opportunity for tangible and visual experiences in this area which can make things more interesting to the student and increase his motivation. Nearly all the experiments done have shown that the children greatly enjoy working with this aspect of mathematics. Facts can be learned through the number line and through geometrical shapes. The learning of the principles of our number system through multisensory experiences with geometry could bring about better results. For example, the teaching of fractions could be greatly enhanced by the demonstration of halves, quarters, etc., with geometric shapes. The child, acquainted with the concepts related to these shapes beforehand, could be aided by learning them in this manner. Experiments by Hawley and Suppes show that it is possible to teach concepts of geometry, and that children can become efficient in constructing and talking about the line segment, perpendicular, bisector, angle, etc. Children have also applied themselves to and seemed extremely interested in learning geometry.

It must be admitted, however, that not everyone is certain about the value of teaching geometry in the primary grades, or even in the intermediate grades. Lamb feels that this subject is not related to the language arts program, and since the curriculum is so concentrated with many subjects already, geometry might not be worth the teachers' efforts [8].* Goldmark finds no practical application of geometry to any other subject studied in the primary grades or to everyday life [4]. She asks, "If geometry is discontinued after the primary grades, has the learning been meaningful enough to be retained? If it continues through the grades, will the concepts become too difficult for children of nine, ten, or eleven years of age?" Both of these writers have had experience with the formal-type presentation of geometry to primary students. Perhaps the quotation by Goldmark might have some validity in this case, although we must never underestimate the capabilities of children. One might ask, does a program need to be so formal in its presentation and concepts?

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* Numbers in brackets refer to the bibliographic entries at the end of the article.

The objectives of any program would depend somewhat on the nature of the program, that is, whether the program was made up of formal or informal experiences. Either type of program would have as one of its overall objectives the development of concepts regarding geometric figures. I. H. Brune suggests that this objective can be met by having the children study and compare the form, shape, size, pattern, and design of geometric figures [2]. They should also have experience in constructing and measuring the figures. R. Briscoe points out that experiences in geometry provide primary children with the opportunity to develop scientific thinking [1]. Briscoe lists four specific contributions of geometry:

1 Children can see space as something they can understand, use, control, and manipulate to their own advantage.
2 They learn the names and properties of the basic geometric shapes and their relation to space.
3 They learn how shapes can be useful to mankind. An example would be triangular supports used in construction in building bridges and buildings.
4 Children learn to produce shapes.

Goldmark adds some other objectives to the ones mentioned above. She points out that children learn to manipulate the ruler, compass, and other instruments used in construction. They derive their satisfaction from the mastery of these skills while they actively utilize the concepts and terms which they are learning. Another very important aspect is that students learn to follow step-by-step directions with precision. Many children tend to be "skimmers"; that is, they learn the general concepts of their various subjects, but they fail to be specific and become careless in their work. Following directions in the construction of various geometric shapes and patterns forces them to follow directions carefully and accurately. Teachers also must be very careful to distinguish the *marks* on the paper which represent a point, line, circle, angle, etc., from the *concept* or idea which each represents. Unless this is done, children will not have a proper understanding of the geometry which they are learning.

Children must be ready to study geometry. Geometric readiness can be reduced to two main factors: subject readiness and psychological readiness. There are several activities which children can do to become ready for experiences with geometry. Handling objects develops the ideas of square corners, straight edges, and curved surfaces. Playing with blocks gives them opportunity to arrange the blocks into patterns. They can cut geometric figures, or learn of line segments, midpoints, and congruency by working with designs.

When beginning instruction in geometry, the teacher should relate the subject to children's everyday experiences. The object of such a presentation is to show the children meaningful relationships rather than make them learn abstractions from memory alone. Having children notice geometric shapes in the world about them can be meaningful. There are many geometric patterns to be observed, including round clocks and tin cans, rectangular chalk boards, perpendicular edges of a sheet of paper, windowpanes, buildings, bridges, church windows, ice cream cones, wheels, sea shells, and telephone dials. In presenting concepts of relationships, the teacher can also use familiar objects such as blocks, balls, and patterns in cloth.

A good time to introduce geometry is during Christmas or another holiday season. Children can use many geometric shapes and figures to make decorations. They can cut out circular, triangular, and square regions, and make pictures by pasting them on another sheet of paper. They can also make three-dimensional objects. Semi-circular regions are easily made into cones, which in turn can be converted into bells, or even Christmas trees. Spirals cut from circular regions are good decorations for the tree. Easter lilies can be made from such shapes. There is no limit to the possibilities. While engaged in these activities, children experience the changes necessary to convert one shape to another—the cut necessary to make a "circle" or a "semicircle," the twist necessary to change a "semicircle" to a cone, the diagonal line necessary to form a triangular region from a square region. As Sweet and DeWitt said, we should show that mathematics, as well as art and music, has aesthetic appeal [10]. I believe these activities will also cultivate the interest of the children.

Children also may enjoy working with subject-free geometric patterns. They could cut out various shapes and make a design with them, or perhaps they would like to make designs by dividing one basic shape into others. For example, Figure 1 shows equilateral triangles divided into other equilateral triangles, and Figure 2 shows other patterns which children can construct from an equilateral triangle.

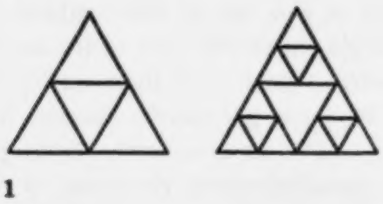

Figure 1

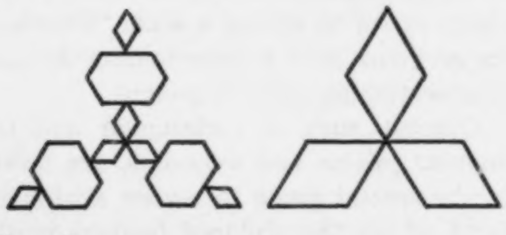

Figure 2

Not only can children create patterns at random, but they can be given problems which call for manipulation to challenge the student. Here are some examples:

1 See how many ways you can divide a square into two congruent (identical) parts (Fig. 3).

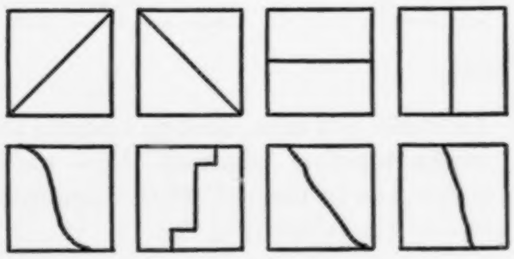

Figure 3

2 See how many ways you can divide a square into four congruent parts (Fig. 4).

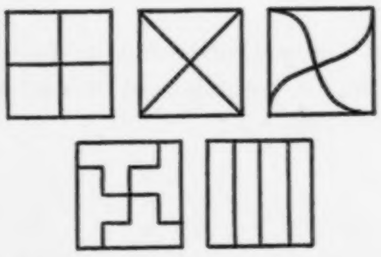

Figure 4

3 How many ways can you assemble three congruent squares? (Fig. 5).

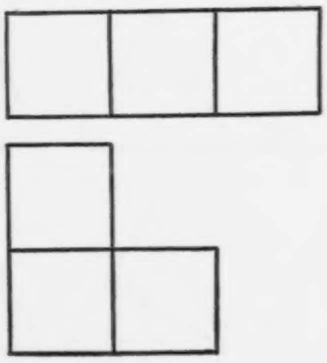

Figure 5

4 Assemble four congruent squares in five ways (Fig. 6).

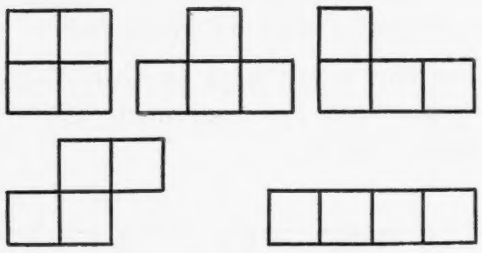

Figure 6

5 Assemble two congruent equilateral triangles together edgewise. How many shapes can be formed? With three equilateral triangles? (Fig. 7).

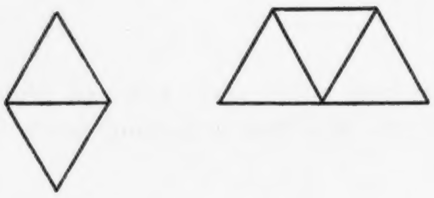

Figure 7

6 How many shapes can be formed by joining four equilateral triangles edgewise? (Fig. 8).

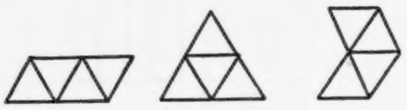

Figure 8

7 Divide an equilateral triangle into three congruent parts in as many ways as you can; into two parts (Fig. 9).

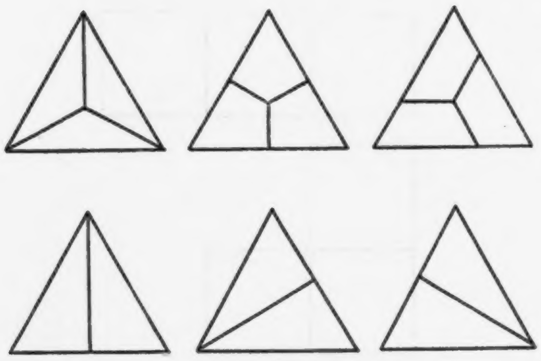

Figure 9

8 Divide the equilateral triangle into six congruent parts; eight parts (Fig. 10).

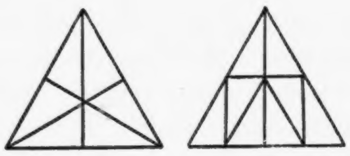

Figure 10

Paper folding is also a good way to present challenging ideas. Some suggest using waxed paper rather than regular paper for teaching because the wax makes a white mark when creased. Some good exercises with paper folding for primary pupils would be a straight line, square corner, bisection, parallel lines, angles, diameter and center of a circle. A pentagon may be made by tying a strip of paper in a knot, tightening and creasing flat, then cutting off the long ends. Children can learn much about symmetric patterns also. They should fold paper to make two perpendicular creases which divide the paper into four equal parts. Then they should fold to bisect the right angles formed, and cut notches out of the paper. All these activities will help them gain insight into concepts of geometry through personal experiences.

Another idea for the presentation of geometry is the use of the pegboard. Pegs can be placed at the apex of the angles of a geometric figure and then outlined with yarn to form geometric shapes. Various figures, such as a square, rectangle, triangle, parallelogram, rhombus, pentagon, and others can be made. The use of brightly colored yarn will also make it easy to demonstrate how one shape is contained in another. This pegboard can also be used to teach measurement, and is especially useful to set up a scale "drawing" for problems such as determining the area of or sectioning a plot of ground.

Objects, such as rectangular and triangular prisms and cylinders, are useful in the second grade to review arithmetic facts which the children learned in the first grade. They can review counting by enumerating the number of sides or edges of a prism. Then, by counting the number of larger sides and the number of smaller sides, the children can add to find the total number of sides. They can subtract the number of edges on a triangular prism from the number of edges on a rectangular prism to see how many more edges there are on the rectangular prism. Here the children can review the facts in a new way, so that they are not bored with the same activity they used in the first grade.

Another activity associated with geometry is work with an array of dots. The children can determine the number sequences as more rows are added. They may enjoy comparing an array in the form of a triangle with an array in a square, both having the same number of rows.

Geometry has other facets besides computation of geometrical formulas. A geometry program for the primary grades should be more exploratory and informal —at least in Grades 1 and 2. All students should be given a purpose for learning these facts; geometry needs application. Children gain reasoning and deductive experiences through work in the application of geometry.

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The role of geometry in elementary school mathematics

G. EDITH ROBINSON *The University of Georgia, Athens, Georgia*

Dr. Robinson is a member of the department of mathematics at The University of Georgia, where her major responsibility is teaching mathematics courses for both elementary and secondary teachers.

Many reasons have been given for the inclusion of more geometry in the lower grades. Certainly, geometry is an important subject in its own right, and as certainly, it is unrealistic to postpone the study of geometry until it can be approached in a systematic and rigorous way. (Precision and rigor can best be appreciated when we understand what we are being precise and rigorous about.) It is also true that geometry is encountered in everyday life, and that children do find the subject interesting. What has not often been pointed out is that geometry can *extend* and *enrich* the study of arithmetic.

Historically, geometry and arithmetic "grew up together." The ancients studied both, and Euclid, in his *Elements*, compiled not only the geometry, but also the number theory known in his day. Thus the separation of the two subjects appears to be an artificial one; actually, they complement one another.

In this article, we will discuss ways in which geometry can add new dimensions to some of the understandings we expect children to gain in arithmetic. We will explore five areas: measurement, properties of the natural numbers, the meaning of fraction, order properties for the natural numbers and rational numbers, and the concept of operation.

Measurement

It is sometimes said that there are two kinds of measurement—that related to discrete quantities (called *counting*) and that related to continuous quantities. In the study of counting, we make frequent use of physical models, presenting collections of pencils, marbles, books, and the like which the child can see and touch. The property of "discreteness" can be made very obvious for the child if he sets aside each object as he counts it or points to each object in turn. But what can we use as models for continuous quantities? One continuous quantity we can measure is time, yet we cannot display any "time" for the child to see and touch. We measure "capacity" but display only the empty container. Even if we show a container full of sand, it is not the sand which we are measuring. Some reflection on the matter suggests that "capacity" is a rather abstract idea.

Geometry provides both models from which a sense of continuity can develop and situations in which "discreteness" can be distinguished from "continuity," provided, of course, that we emphasize these properties. Consider, for example, a path joining two points in a plane. Such a path is said to be *connected:* we can draw a picture of it without lifting our pencil from the paper—the pencil moves *continuously* from one point toward the other. Figure 1 illustrates two such curves.

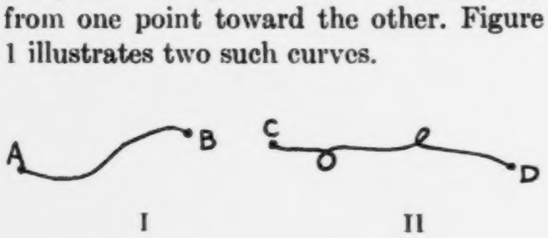

Figure 1

We think of each of these curves as being "*all in one piece.*" We also think of each of them as being *a set of points*. These represent two fundamentally different interpretations. We might say that one interpretation is dynamic, the other static. To emphasize the difference in these interpretations and the consequences arising from this difference, we might raise the following questions:

1 Suppose we remove one point from Curve I; will the curve still be "all in one piece"?

We see, of course, that the answer depends on our choice of the point to remove; if we choose either point A or point B, the connectedness will be undisturbed; selection of some other point of the curve will separate the curve into two pieces. Each of the two separate pieces, however, will retain its connectedness.

2 If we remove one point from Curve II, will the curve still be "all in one piece"?

Again we see that this depends upon our choice of the point to remove; as in the first case, removal of either "endpoint" (C or D) will not disturb the connectedness property. Unlike the first case, however, it is possible to remove a point other than an endpoint without separating the curve (try a point on one of the loops).

3 Is a proper subset of Curve I ever connected?

Looking back at question 1, we see that a proper subset of a connected set does not *have to be* connected. We can see, however, that some proper subsets *may* be connected. Suppose we select some third point X on Curve I and look at the subset of the curve consisting of X, A, and all points of the curve "in between." Such a subset is clearly connected (as, of course, would be true of the subset consisting of X, B, and all points of the curve "in between"). Thus we see that a proper subset of a connected set may or may not be connected.

Another example of a proper subset of Curve I which is not connected is the subset consisting of the points A, X, and B. This, of course, is a *discrete* subset, and we note that a connected set may have proper subsets which are discrete or connected (or neither).

The property of being "all in one piece" is easily recognizable, and we need not try to make it more precise. As a matter of fact, preschool children may sense such a property in an object even before they sense its distinctness (or "oneness"). However, the school experiences accompanying the study of counting may have disrupted the perception of connectedness, and it may be pedagogically wise to introduce experiences in which the child can learn to discriminate between "discrete" and "connected" sets. As we have just seen, geometry provides appropriate experiences of this sort.

Extending our discussion, we can consider closed curves—paths from one point to another and back again. Again we consider the two examples in Figure 2.

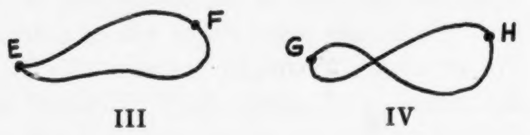

Figure 2

Here also we have sets of points, and sets of points which have the property of being "all in one piece." The type of curve we have in mind is one whose picture can be drawn without lifting the pencil from the paper. In relation to these curves we can also raise some questions:

4 If we remove one point from Curve III, is the curve still connected?

We see a difference between this "closed curve" and the curves in the previous examples—it is impossible to separate this curve by removing a single point.

5 Can Curve IV be separated by removal of one point?

There is exactly one "separating point" for this curve; unless we remove the point where the curve "crosses itself," the resulting set will still be connected. Is it possible to have a closed curve with more than one separating point?

6 Is a proper subset of Curve III always connected?

Our answer to question 4 tells us that a proper subset of Curve III *may* be connected. To see that a subset need not itself be connected, we need consider only that subset of Curve III which results from removing two points, or the discrete set consisting of two points of the curve. Students might think that a set of two points "right beside each other" would still be connected; this gives us an opportunity to emphasize that there is no such thing as "the very next point." We do not draw the picture by drawing a "line of dots" because such a procedure does not conform to our idea of what a "path" is.

We have looked at some sets of points which we have called "paths," and have discussed the idea of connectedness. Now we will examine some other sets of points. For example, we might consider the line, the plane, and space itself. "Space" is defined as the set of all points. Each of these sets of points is considered to be connected.

Following the same procedure as above, we might examine subsets of each of these. Some subsets of the line are the segment, the half-line, and the ray. Are these connected sets? We could also pick a subset of a line which consisted of three points (or four, or two, etc.); would this subset be connected?

We have already looked at some subsets of the plane (Curves I–IV and the line), but there are many others. For example, there is the subset of the plane interior to Curve III, and the subset of the plane exterior to Curve III. Then there is the angle, the exterior (and interior) of the angle, and the half-plane. Each of these is a connected set. If we look for subsets of the plane which are not connected, we can think of any discrete set of points, or a subset such as the plane with some closed curve (a triangle, for example), removed.

Some subsets of space which we have not already considered are the solids and the portions of space bounded by solids. We might use the cube as an example. Our idea is that the cube is a connected set, and we see that the subset of space interior to (or exterior to) the cube is also connected. Some subsets of space which would not be connected would be any discrete collection of points, or a subset such as space with a cube removed.

Our purpose so far has been to emphasize the difference between "discrete" and "connected" sets, since geometry provides us with examples of both. In looking at the various examples, however, we can scarcely avoid laying the foundation for measurement in the geometric sense. One feels intuitively, for example, that Curve II is "longer than" Curve I, that the piece of the plane enclosed by a loop of Curve II is "smaller than" that enclosed by a loop of Curve IV, and that some cubes are "larger than" others. To make such intuitive ideas precise, we *measure* the lengths of curves, *measure* subsets of the plane interior to closed curves, *measure* solids, and *measure* subsets of space interior to solids. In other words, the terms "perimeter," "area," "surface area," and "volume" are names applied to the measurement of connected sets. We note also that we choose a connected set as a unit of measurement in each case.

Properties of the natural numbers

Geometrically, a natural number, N, which is greater than one is a composite if any collection of N objects can be arranged in some rectangular form; that is, if the N objects can be put into rows and columns, with both the number of rows and the number of columns being greater than one. Twelve, for example, is composite, since we can arrange twelve objects in any of the

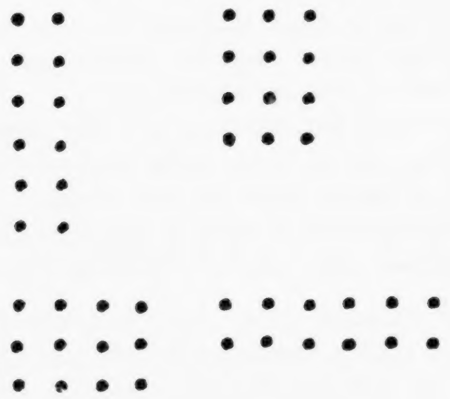

Figure 3

ways shown in Figure 3.[1] On the other hand, seven is a prime number, because no matter how we try, we cannot arrange seven objects into rows and columns (except one row or one column) without having "one left over."

If, in working with our N objects, we can put them into a *square* array (equal numbers of rows and columns), we say that N is a "perfect square."

Many teachers illustrate the commutative property for multiplication by placing objects into rectangular arrays in two ways; what is suggested here is that the *children* experiment with collections of various sizes, determining *for themselves* which numbers are "rectangular" and which are not. Depending on when such an activity is initiated, it could serve either to introduce the multiplication facts or to reinforce the learning of these facts.

Another geometric activity which can aid in the understanding of the properties of the natural numbers begins with a collection of square-shaped regions. Suppose we cut some of these, all the same size, from a piece of paper. We can ask the children if they can put several of these together to make a larger square region. As they experiment, it becomes obvious to them that they cannot do this by putting two pieces together, or three. Four pieces, however, can be put together to make a larger square region (Fig. 4).

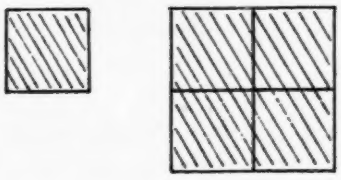

Figure 4

Next, we ask the natural question: Can more pieces be put together to make a still larger square region? Experimentation shows that this cannot be done with 5, 6, 7, or 8 squares; the next larger square requires nine small ones. Continuing the activity, we see that we require successively 1, 4, 9, 16, 25, etc., small squares to make a larger one having the same shape.

Next, suppose we take triangular shapes (Fig. 5) and raise the same question: Can we put some of these together to make a larger one of the same shape? Through trial, we find that two will not do, or three; again we find that a larger triangular region requires at least four small ones (Fig. 6).

Figure 5

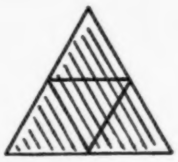

Figure 6

(In case a child might think that three will do, we emphasize that the interior of our figure must be *connected*. See Figure 7.)

Proceeding, we discover that it requires successively 1, 4, 9, 16, etc., small regions to build a larger region of the same shape. To extend geometric understanding, we

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[1] Some may object to calling this discrete collection "rectangular." However, it may be argued that the viewer can impose the "rectangular" interpretation on this array only if he understands the connectedness property.

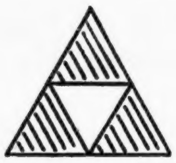

Figure 7

might ask if other shapes would give the same results; for example, can we build larger rectangles from smaller rectangular regions? What about parallelograms, or circles? And, if so, how many will be required in each case? We might also extend the activity to cubes: How many cubes must be assembled to make a larger cube?

Concept of fraction

The "fraction pie" is a familiar device for the introduction of fractions, and is very convenient to use because of its symmetry. Geometrically, of course, the fraction pie is not a pie at all, but a circular region, a subset of the plane. In using it as our unit, we are appealing to its property of being connected. In other words, when we begin the study of fractions, we select a unit to partition, and we choose for our unit *something which is connected.* To develop understanding of the meaning of fraction, then, suppose we choose for our unit some other connected set. We might consider any of the sets in Figure 8. Questions relevant to the meaning of fraction which we could raise for each of the units in Figure 8 are:

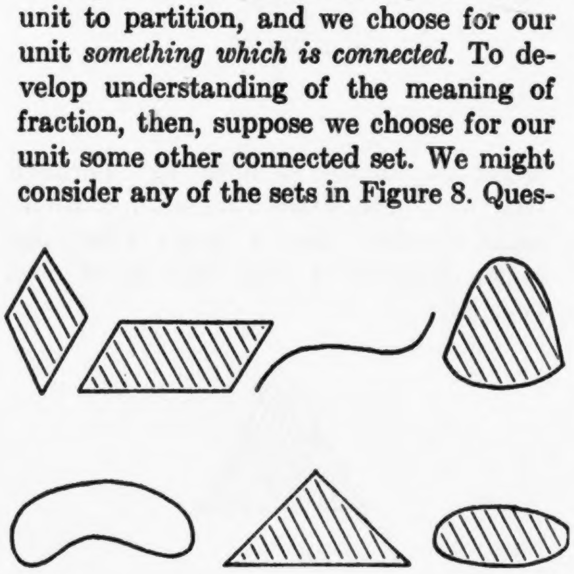

Figure 8

1 Can each of these be partitioned into halves with certainty?
2 Is there more than one way in which these units can be partitioned into halves with certainty?
3 If there is more than one way to partition one of these units into halves, in how many ways can this be done?
4 Can the above units be partitioned into thirds with certainty? Into fourths?

A question which we must anticipate from such a discussion is: Must the parts of the unit be the same *shape* to be the same *size?* A case in point is illustrated in Figure 9, each piece shown being one-half of a half.[2]

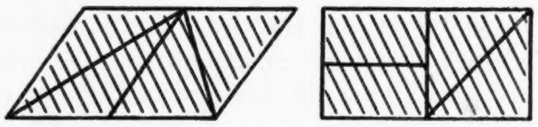

Figure 9

Since either a closed curve or the region bounded by that curve might be selected as a unit, we might examine the diagrams in Figure 10. After studying the diagrams, we might consider these questions:

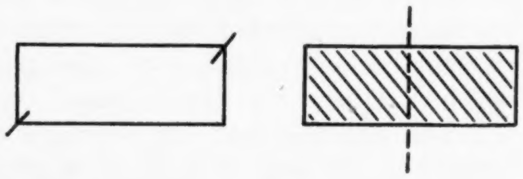

Figure 10

5 If a region is partitioned into halves, is the curve bounding that region necessarily partitioned into halves?
6 If a closed curve is partitioned into halves, must that part of the plane interior to the curve necessarily be partitioned into halves?

Also, while we are extending ideas, we need not confine ourselves to plane figures.

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[2] It has been determined that third-grade children can find as many as twelve different ways of partitioning a rectangular region into fourths.

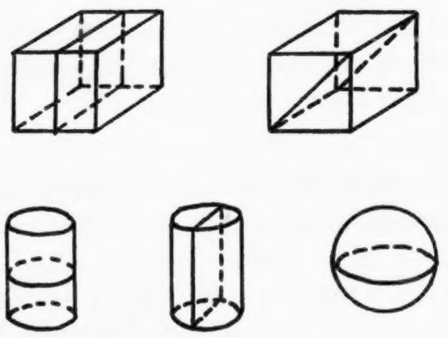


Figure 11

Five solids are illustrated in Figure 11. Are there ways of partitioning the solids other than those illustrated? In how many ways can each of these solids be partitioned into fourths with certainty? Can we devise ways of partitioning these solids into thirds with certainty?

Properties of order for the natural numbers and rational numbers

When we count, we say the word "five" immediately before we say the word "six." That five "comes before" six is not merely convention—notice that it is not the same as saying that "*a* comes before *b* in the alphabet." Rather, the statement "five comes before six" expresses the mathematical fact that five is *less than* six. We can show this to children with discrete sets of objects; we can also show this to children geometrically with a number line.

When we looked at curves before, we stressed that the property of being connected was related to drawing a picture of the curve without lifting the pencil from the paper. As we draw, we get to certain points in the curve before we get to others. In Figure 12, if we draw the path from A to B, we get to point X before we get to point Y. This fact is so obvious that the reader may wonder why we bother to mention it. We mention it because the very fact that this is so obvious may mean that it is easier for children to understand what is meant by "five is less than six" with the aid of the number line.

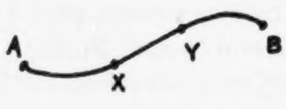


Figure 12

For the natural numbers, we actually use a ray rather than a line; in drawing Figure 13 we get to point X before we get to point Y.

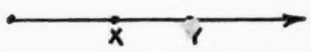


Figure 13

For the natural numbers, we have not only order, but also order in a very particular way: 2 is 1 greater than 1, 3 is 1 greater than 2, etc. Thus on our number line we do not select points at random, but make use of a *unit*, and lay off this unit in a regular fashion, beginning with the endpoint of the ray (Fig. 14). Since point A "comes before" point B, and since B is *one unit* to the right of point A, it is natural to assign the natural number 1 to point A, the natural number 2 to point B, and so forth (Fig. 15).

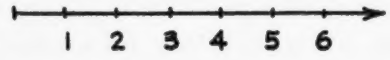


Figure 14

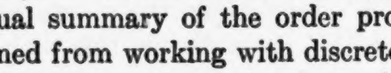


Figure 15

The number line gives a more "panoramic view" of the very regular ordering of the natural numbers. For some children, this may be easier to understand than considering order in relation to discrete sets of objects; for other children, it may provide a visual summary of the order property obtained from working with discrete sets.

Order among the rational numbers is also important. Since we established a *unit* to construct our number line, we can then assign "fraction names" to other

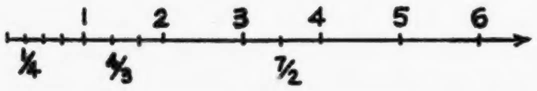
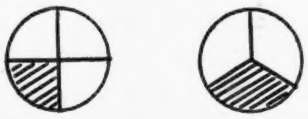


Figure 16

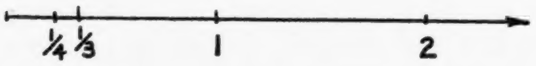
points on the ray (Fig. 16). Then we might think about which of the following is more helpful in understanding that "$\frac{1}{4}$ is less than $\frac{1}{3}$":

(*a*) "$\frac{1}{4}$ is less than $\frac{1}{3}$ because the more pieces into which a unit is partitioned, the smaller those pieces must be."

(*b*)



(*c*)



Is it possible that different readers might make different choices from these three? Is it not possible that different children might find one of these more meaningful than the others? Is it not possible that one of these "explanations" would be more meaningful for the beginner (in the study of fractions), whereas another would be meaningful after fractions had been studied for a while?

Sometimes we make statements about order which say more than simply "5 is less than 6"; we may say, for example, "5 is less than 6 and greater than 4," or "$\frac{1}{3}$ is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$." Algebraically, we write these "$4<5<6$" and "$\frac{1}{4}<\frac{1}{3}<\frac{1}{2}$." At other times we say "5 is between 4 and 6," and "$\frac{1}{3}$ is between $\frac{1}{4}$ and $\frac{1}{2}$." These latter versions really express what we see (geometrically) on the number line—namely, that the point designated "$\frac{1}{3}$" lies *between* the points named "$\frac{1}{4}$" and "$\frac{1}{2}$." If we discuss this with children, they may ask if every point on the number line has a fraction name. This gives the teacher the opportunity to explain that such is not the case, and that other number systems will later be studied which do provide number names for all points on a line.

Extending the concept of operation

We can demonstrate to children that $2+3=5$ with discrete sets of objects—that is, we can join a set of three pencils to a set of two pencils and determine the numerousness of the union. We can also demonstrate that $2+3=5$ with connected sets on the number line. To do this, we think of "taking a step of size 2" followed by "a step of size 3." To represent this pictorially, we let arrows represent the step—a step of size 2 is represented by an arrow 2 units long, a step of size 3 by an arrow 3 units long. "Followed by" is represented pictorially as being laid end to end (Fig. 17). The result can be seen (literally) to be equivalent to an arrow 5 units long.

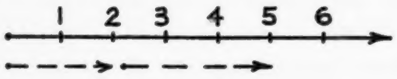


Figure 17

Furthermore, this *interpretation* of addition applies more readily to addition for the fractions than the concept of the joining of sets. Although it may not be easy to see what the sum of $\frac{2}{3}$ and $\frac{3}{4}$ is through the use of arrows, at least this interpretation may be shown, whereas it is a little difficult to show what we mean by joining $\frac{2}{3}$ of a pie to $\frac{3}{4}$ of a pie. Actually, in the long run, this geometric interpretation appears to be more efficient; when faced with addition for the integers, for example, it is difficult to interpret $(+2)+(-3)$ in terms of the union of two discrete sets!

Multiplication can also be interpreted geometrically: 2×3 can be interpreted as "2 steps of size 3," and can be represented pictorially by two arrows, each three units long, placed end to end. Furthermore, it is easy to *interpret* subtraction as the inverse of addition, and division as the inverse of multiplication within this geometric context. Such an interpretation, like that for addition, is appropriate for these processes in the system of integers. This, of course, is what we mean when we speak of "extending" the concept of operation—providing a model which has wider application than the first one studied.

Conclusion

We do not know with any degree of certainty that all children learn in the same way. We know that all children do not learn at the same *speed*, and strongly suspect that they do not learn in the same *way*. Thus a meaningful explanation to one child may leave another "in the dark." Knowing this, experienced teachers use a variety of ways in which to present arithmetic ideas to children. We have suggested here that geometry has a contribution to make to the extension and enrichment of certain understandings of arithmetic. The very fact that we need pictures to demonstrate geometric ideas means that we have visual models for the associated ideas of arithmetic; all we need to do is emphasize the connection between the two.

Nor should we overlook the unique contribution which geometry can make to arithmetic: certain sets of points have the property of connectedness. This property of being connected is one which is closely related to the concept of fraction, and is basic to the concept of measurement. Moreover, it can be used to impart new meaning to concepts such as those of operation, and of order and denseness among elements of a number system.

Geometric concepts in Grades 4–6*

DORA HELEN SKYPEK *Emory University, Atlanta, Georgia*

Mrs. Skypek is a member of the faculty in the division of teacher education and in the department of mathematics at Emory University.

In trying to decide what to say about this subject with only forty-five minutes in which to say it, I asked a colleague for suggestions. He said, "Refute, if you can, the charge of 'pseudosophistication' currently being made against much of what's new in the elementary school mathematics curriculum."

I looked carefully, then, at units on geometry in some of the contemporary textbooks and could easily imagine many elementary school children who, anxious to please their teachers, could and would memorize all the new words, the collections of pictures that go with the words, and the symbols and measurement formulas that go with the pictures, just as children before have memorized their way through arithmetic without really understanding the ideas represented by the language and the mathematical symbols.

While I deplore the charge of pseudosophistication, I concede that it could be true. It *is* possible to teach a mathematically sophisticated language and symbolism without realizing the basic purpose of all our efforts: that children come to possess mathematically sound and mathematically rich concepts. Before deciding *what* geometric concepts to teach and *when* to teach them, we need to consider what research has to say about *how* children learn geometric and topological concepts.

The major contemporary contributor to research in concept formation in the child is Jean Piaget, the Swiss psychologist, philosopher, and educator. Piaget has conducted and published an amazing number of studies of intellectual development and has been largely responsible for many related studies by other psychologists and learning theorists.

At the grave risk of oversimplifying Piaget's many-faceted theory, I have chosen to examine the implications of two of his hypotheses for the teaching of geometric concepts in the upper elementary school.

Much of his theory is dominated by the hypothesis that action-involvement is the key to progress in concept development. With regard to spatial relationships, Piaget's hypothesis is even a bit startling. He emphasizes that action on objects in the child's world, rather than perception of the objects, is of primary importance. John H. Flavell, who is perhaps the most important American interpreter of Piaget's developmental theory, points out that it just seems natural to us to assume that we see space as it is and to assume that we have always seen it that way. Not so, says Piaget. This effortless seeing is really the end product of long and arduous developmental construction, and the construction is more dependent on actions than perception *per se.* One key implication, then, is this: *action on objects precedes perception* and, of course, conception.

Piaget's research led him to the conclusion that concepts involving topological relations precede those of projective and

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* Adapted from a speech made at the Atlanta Area Meeting of NCTM, November 20, 1964.

Euclidean relations. Moreover, topological relations constitute the foundation on which the others are constructed. The implication here is one of order in the development of geometric concepts.

Topological relations of an elementary nature included in all contemporary curricula are these: the distinction between open and closed curves; the ideas of boundary, of regions interior to a closed figure and exterior to a closed figure; the idea of betweenness on open figures. Piaget found these relations to be understood by the early school-age child. That this is so seems reasonable when one considers the following instances of action *on* objects in the child's world.

Little boys scratch out rings in the sand or clay in schoolyards for their games of marbles (Fig. 1). Marbles lying wholly within the ring represent points (A and B in the figure) inside, or interior to, the closed curve; marbles lying outside the ring represent points (C and D in the figure) in the exterior region; and marbles lying on the ring represent points (E in the illustration) *on* the curve, or boundary.

Do little girls still play hopscotch? Children who sketch with sticks in the sand or with crayons on the sidewalk the union of rectangles, as shown in Figure 2, are also action-involved with elementary topological notions. According to the rules of the game, one tries to toss a stone or token of some sort so that it falls in a region interior to one of the closed figures (point A in Fig. 2). However, it sometimes falls on the boundary of a closed figure (point B), sometimes on the common boundary of two closed figures (point C), and sometimes in the exterior region of the union of the rectangles (point D).

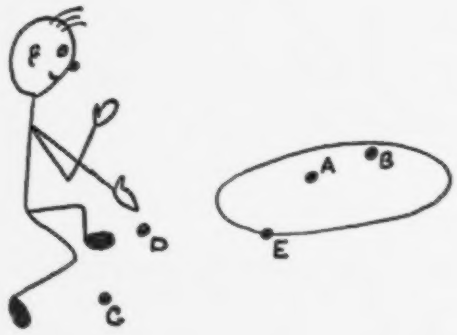

Figure 1

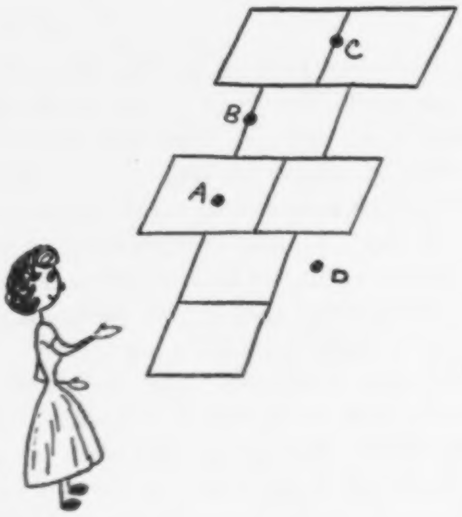

Figure 2

The conceptualizations of simple closed figures, regions interior to closed curves, regions exterior to closed curves, and boundaries of the regions are already "working" concepts for the child who has engaged in these or similar games. The young child can also discriminate between simple closed and open curves, for he travels many paths or pushes things, such as marbles, miniature cars, or doll carriages, along paths—activities which provide the action-involvement from which he conceptualizes the distinction between open and closed curves, points between other points, endpoints, etc.

Consider the example represented by the open curves in Figure 3. Let's imagine that Tom occupies a desk on the opposite side of the classroom from the pencil sharpener. Let point A represent Tom's desk and point B represent the pencil sharpener. Tom has several choices for the path from his desk to the sharpener: (1) He can travel the most direct path from A to B (Fig. 3*a*); (2) he can go along the aisle by his desk to the back of the room, across the back of the room, and

Figure 3a

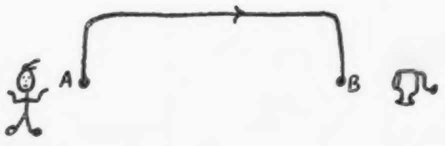

Figure 3b

Figure 3c

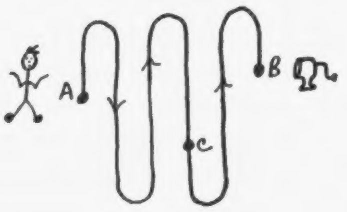

Figure 3d

down another aisle to B (Fig. 3*b*), or (3) he can go to the front of the room and across (Fig. 3*c*). If Tom has complete freedom of choice he may even travel a path up one aisle and down the next (Fig. 3*d*). In fact, he may deliberately choose this path so that (by chance, of course!) he jostles his friend Jack's desk (at point C), which lies *between* points A and B.

Through discussion and questioning, children can be guided to organize their ideas about topological relations by relating their experiences to imaginative representations of them; it is *after* a child acquires working concepts that it is meaningful to provide him with the more formal language and representations to associate with the generalized concepts.

Another topological property included in contemporary programs is that of a continuous infinite set of points. Piaget's experiments indicate that a child's conceptualization of a continuous figure as a connected series of points, infinitely small and infinitely many in number, is again the result of developmental change, and is not usually available to the child before early adolescence.

There is need for more definitive research on this and other topological properties included in elementary school programs. In the meantime, let us be careful to avoid pseudosophisticated glibness about "continuous lines" and "infinite sets of points" without some assurance that the concepts with which this language is associated are suitable ones for the children in our classes. Our task is to recognize patterns of developmental change, a kind of rhythm in intellectual growth, and to "fit" learning experiences and instruction into this rhythm.

Piaget's experiments involving concepts in projective geometry include studies of those properties which remain perceptually invariant under changes in the point of view from which a figure is looked at, i.e., *spatial perspective.* He describes a task in which children were asked to arrange sticks in a straight line. The youngest were unable to do so. The next age group could do so, if the arrangement followed a course parallel to the straight-line edge of a table; otherwise, the arrangement tended to drift toward some nonlinear reference curve. However, the seven-year-olds generally tested for straightness by sighting along the array of sticks from an end-on position.

Of course, the arrangement of objects in a straight line is not of particular relevance for us, but Piaget interprets these results as the child's growing awareness of the existence of points of view and the choice of perspectives for assessing the problem at hand. This interpretation is of significance to us in the development of perspective with regard to two-dimensional and three-dimensional space.

Children in the upper grades need to handle, or "act on," physical models of simple open and closed curves and simple open and closed surfaces which can be looked at from various points of view. For example, let them view a large square tile at some distance from them so that all points on the boundary (the square) are approximately the same distance from their eyes—their "point of view." Let them draw the boundary as it appears to them. The drawing should be the same as the one we usually see representing a square (Fig. 4*a*). Now rotate the tile so that it still lies in the same plane as before, but the vertices of the square do not occupy the same points; have the children draw the boundary as it now appears to them (Fig. 4*b*). Now tilt the tile so that it is no longer in the same plane as before, except for one of the edges; again ask the children to draw the boundary as it appears to them (Fig. 4*c*). Perhaps they can find pictures of a tiled floor in which the "squareness" of the tiles is distorted because of the perspective from which the picture is made (Fig. 4*d*). At this point I can't resist a reference to the rich possibilities of discussions on "Geometry in Art" with special attention to perspective.

Provide worksheets with representations of plane figures as they might look from many different points of view and let children match the representations of similar or congruent figures.

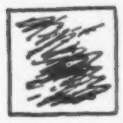

Figure 4a

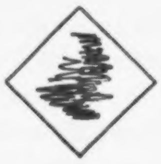

Figure 4b

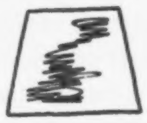

Figure 4c

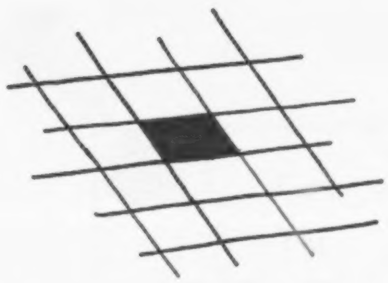

Figure 4d

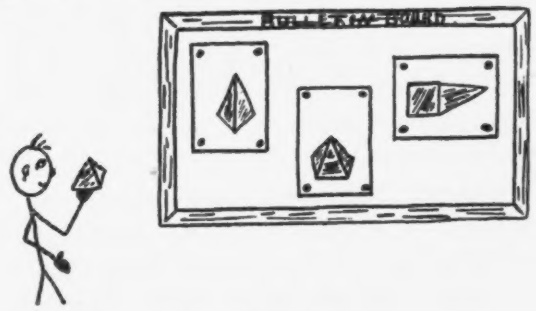

Figure 5

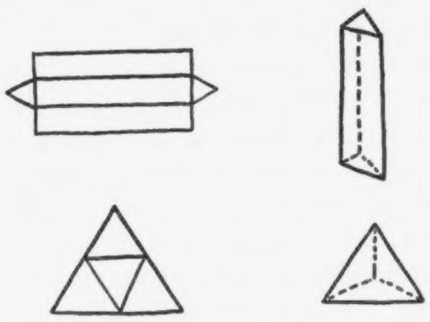

Figure 6

In later experiences, display on a bulletin board drawings of three-dimensional models as they might appear from several different points of view, as illustrated for a pyramid in Figure 5. Let children handle three-dimensional models (they can easily be made from construction paper) and try to hold their models in the positions corresponding to the points of view from which each of the drawings was made. Such an experience also provides an excellent time for using the language of "vertex," "edge," "face," and for discussing the ideas of a closed surface, of a region interior to a closed surface, of a closed surface as a boundary of the interior region, and of a region exterior to a closed surface.

Still later, let the children draw a cube, a pyramid, or another three-dimensional model as it would look if it were "undone" and opened out flat. Or let them match the three-dimensional models with two-dimensional paper patterns for making models of their own. Let them match the patterns for the models with corresponding drawings of the completed models (Fig. 6). Such experiences are also important for many later and more formal learning experiences in mathematics.

In Piaget's treatment of concepts of Euclidean geometry, his studies are primarily concerned with conservation and measurement of length area, and volume. In these experiments he also finds a developmental trend according to age. Since the idea of conservation is central to much of Piaget's theory of mathematical concept development, let me recount one experiment on conservation.

The experimenter and the child have two balls of clay, both of the same shape and size. The shape of one ball is then changed by rolling it into a sausagelike shape, or flattening it into a cake, or breaking it into pieces, while the other ball of clay retains its original shape for comparison. The experimenter tries to find out whether the child thinks the amount, the weight, and the volume of the clay have been changed by the transformations, or whether he thinks they remain unchanged, i.e., whether the amount, the weight, and the volume of the clay have been *conserved.*

Piaget's subjects indicated working concepts of conservation of matter regardless of perceptual changes at ages 8 to 10, conservation of weight at ages 10 to 12, and conservation of volume at 12 years or later. The important implication here is *not* that there is a "natural" maturation process going on with increasing age and that all we need to do is wait until the child ages sufficiently—we know better than that! Instead, the important implication is this: there is a pattern of sequential stages which mark concept formation. Learning experiences and teaching techniques need to be devised to provide active participation by the children, exchange of ideas with other people through class discussions, and refinement of the child's own developing processes according to a developmental sequence.

For instance, in working with the measurement of area, before any formal generalizations or computational rules for finding area are taught, we might provide experiences such as the following, in which children "act on" the plane regions to be measured and discover for themselves the idea of conservation of area.

First, we will assume that the child already possesses the concept of a linear unit for measuring line segments or the union of line segments. Let me digress still further and summarize the fundamental notions involved in the concept of measurement:

1 The measurement process requires the choice of a unit that is of the same nature as the thing one wants to measure.
2 The unit needs to be such that it can be moved around or copied for comparison with the thing one wants to measure.
3 Measurement is a process of assigning a number to a set or an entity of some kind and yields an approximate number of units.
4 In measurement, one chooses an appropriate unit.

Now I shall use representations of rectangular disks; however, in the classroom one should begin with actual models of rectangular regions, such as table tops, bulletin boards, or rectangular pieces of cardboard which the child can "act on." The figures in Figure 7 represent a rectangular region. To measure the surface one must choose (1) a unit of the same nature as is the thing to be measured, i.e., a unit possessing surface; (2) a unit that can be moved around for comparison with the surface to be measured, and (3) an appropriate unit. One possible unit is a circular disk (Fig. 7*a*). It has surface, and

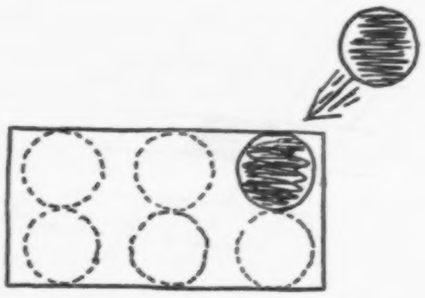

Figure 7a

so is of the same nature as the thing to be measured. It can be moved around or copied for comparison, but it is not convenient or appropriate—i.e., six circular units of surface compared with the surface of the rectangular region do not even closely approximate the surface of the region. Another possible unit is a triangular disk (Fig. 7*b*). It has the same nature as the thing to be measured, and it can be copied and moved around. Although it is a better choice than the circular disk, it is not so convenient as a square disk (Fig. 7*c*). Measurement also involves the process of assigning a number to this plane rectangular region. Thus, we *count* the number of square disks it takes to cover the region. We say the measure of the plane region is 6; there are 6 units of surface measurement in the region.

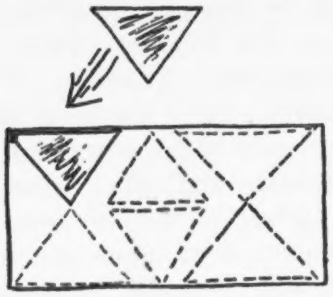

Figure 7b

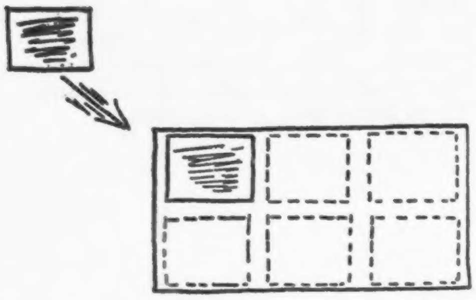

Figure 7c

Consider the rectangular region represented in Figure 8*a*. The rectangle enclosing the region measures 3 linear units by 4 linear units. What is the measure of the interior region? In other words, what is the area of the rectangular region?

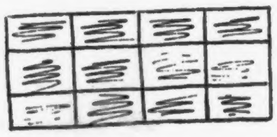

Figure 8a

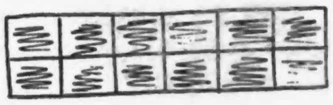

Figure 8b

There are 3 rows with 4 square units in each row, or 12 square units. Notice that the area of the rectangular region in Figure 8*b* is also 12 square units, or 2 rows with 6 square units in each row. The area of the two rectangular regions is the same, even though the rectangles are not congruent; furthermore, the perimeters of the two rectangles are not the same. The first rectangle has a perimeter of 14 linear units; the second rectangle has a perimeter of 16 linear units.

In these first experiences, let the children handle the square disks; let them "act on" the thing to be measured by placing enough square disks on it to cover the region, and then let them *count* the number of square units of surface in the region. Also, let them count the number of linear units along the boundary, or rectangle. The words "area" and "perimeter" should come *after* the children grasp the idea of square units for measuring plane regions and linear units for measuring line segments or the union of line segments (as in rectangles and other simple closed linear figures).

Consider the rectangular region represented by Figure 8*c*. Do you suppose the area of this region is the same as that of the other two? An attempt to use the same unit of measurement poses a problem. Shall we use a smaller unit? To compare the areas, we need to use the same unit.

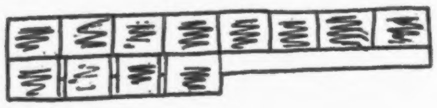

Figure 8c

Give your children an opportunity to discover the solution to this problem. They have few inhibitions about cutting some of the square disks into halves and making a "fit." And, again, although the perimeters are not the same, the areas are.

The foregoing examples are all rather elementary. The point I seek to make is that experiences in which the child "acts on" the objects in his world rather than passively observing someone else, usually the teacher, showing and telling him about geometric relations and properties is a *first* step in the developmental formation of geometric concepts. The child who has these experiences is less likely to confuse the concepts of perimeter and area than those who, without action involvement and small group "discovery" experiences, simply memorize computational rules for determining perimeter and area.

Of course, in still later learning experiences, children must learn that society has adopted certain *standard* units of measurement, both the English system and the metric system. Conversion ratios from, say, square inches to square feet, and vice versa, will have real meaning for children who have "acted on" the problem of how many square inches are in a square foot by placing 12 rows of 12 cardboard square disks (each disk representing a square inch) on a larger cardboard square disk which measures one foot on each side, and then *counting* the number needed to cover the larger disk.

In closing, I encourage you to experiment, to conduct your own action-research in your classrooms to determine what geometric concepts should be included in the upper elementary school mathematics program. Read the research report in the October, 1964, issue of THE ARITHMETIC TEACHER by Charles D'Augustine. He reports a study on teaching topics in geometry and topology to an average sixth-grade class. In the final paragraph he calls for more research to determine what topics are teachable, suitable, and efficiently learnable at the various levels of the elementary school. The criteria of teachability, suitability, and learnability are worth remembering.

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Geometry in the grades

IRVIN H. BRUNE *Iowa State Teachers College, Cedar Falls, Iowa*
Dr. Brune is professor of mathematics at Iowa State Teachers College.

The proper study of mankind is geometry. Tiny Timothy chose a nickel rather than a dime, but it was worldly wisdom, rather than geometric sense, that he lacked. Timothy's grandfather, however, who was shrewd in money matters, believed that three-foot cubes rather than three-foot spheres should adorn the new courthouse. He surmised that the cubes, covered only on five faces, would require less gold leaf to gild. His geometric guess was almost as naive as Timothy's money muddle.

Myra in Grade 5 wanted to grow flowers. Her parents gave her fifty feet of aluminum edging and told her that she could have all the ground that the edging would enclose. What shape of flower bed would give her the most space?

At a sale Mr. Handyman bought a piece of linoleum that was nine feet wide and sixteen feet long. He knew that with only one cut he could fit it on a floor twelve feet by twelve feet. This, as well as the other cases cited, requires geometry.

Or consider the circles in Figure 1. Obviously the black circle on the right is larger than the black circle on the left. But is it?

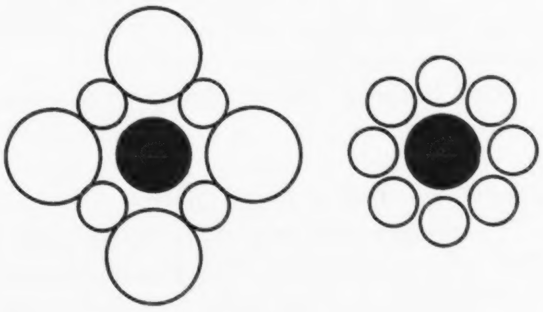

Figure 1

Similarly for Figure 2. Which has the greater area, the outer black annulus (or ring) or the inner black circle?

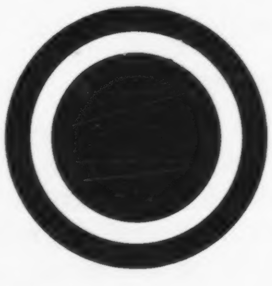

Figure 2

Another instance is the well-known Figure 3. Which segment, the horizontal or the vertical, is the longer?

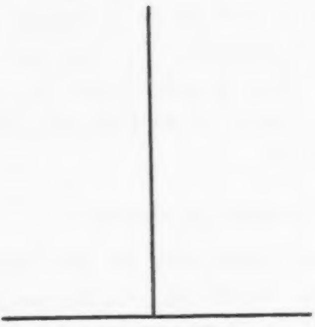

Figure 3

Perhaps you think such examples are trivial. At least they are homely, ordinary. But there are more important applications too; you see them daily. When mathematicians use higher geometry to solve complicated problems in the production and distribution of goods, no one concerned thinks the matter trifling. Consider, for example, the problem of selecting the most economical combinations of some twenty ingredients that fluctuate in price daily. This problem arises in sausage-making. Which quantities of which meats will satisfy fixed standards of high quality

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and at the same time cost the least according to today's prices? Here the mathematician employs geometry in an untrivial way.

Ageless geometry

To enumerate and to describe man's uses of geometry would take a trip in time from prehistory to the present moment. The subject began in earth-measuring, it grew in planet-observing, it led the way in pure mathematics, and it pioneered in modern mathematics.

Man has always needed geometric principles, however dimly he may at first have perceived them. Similarly, children's lives cannot be devoid of geometry, however unaware they may be of its formal aspects. For, irrespective of its many applications and regardless of its value as a system of reasoning (and both of these phases merit attention), geometry embodies numerous ideas interesting in themselves.

Geometry for all

We suggest, therefore, that geometry deserves a lifetime of interest. To study it in only the tenth grade hardly suffices. At that level pupils presumably study one or more kinds of geometry as deduction. There and in subsequent courses they also learn about applications. But the computing with geometric formulas that frequently represents the only planned experience that pupils have in geometry prior to Grade 10 seldom prepares them for Grade 10.

Grade-school geometry

Informal geometry in the elementary grades can, therefore, counteract a serious deficiency. In these grades geometry is the study of form. Shapes, sizes, patterns, designs—these are the stuff from which children form concepts. From studying forms children discover numerous geometric relations; from making constructions pupils learn about geometric facts; from measuring figures learners acquire a background of geometric information. The work teems with both classical concepts and contemporary concepts.

Figure 4

Fun and future

We believe, therefore, that children of all ages should get ample opportunities to find out things about geometry. The goal is satisfaction, here and now, with things mathematical, and geometry abounds in such ideas. An accompanying benefit will doubtless be a preparation for a more formalized geometry of proofs. Just as the ancient Egyptians' surveying and the early Chaldeans' star-studying opened a path for the deductions of the Greeks, so the block-arranging of the curious kindergartners and the design-drawing of the enthusiastic upper graders provide understandings for the problems of the older pupils. The pleasures of the moment outweigh the preparations for the future.

Therein lies the heart of the matter. Teachers cherish in their pupils such traits as alertness, preparedness, and willingness. And possibly the greatest of these is willingness. Seldom, though, do these traits develop overnight; rather, they seem to stem from many things that pupils do. Through the situations that teachers encourage them to explore, pupils discover relations, achieve insight, and gain satisfactions for the moment as well as for later studies. Mathematics, you know, is a cumulative subject. For example, clusters of dots, such as those in Figure 4, provide numerous helpful experiences. For infants the dots in Figure 4 are many, whereas those in Figure 5 are few.

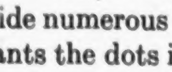

Figure 5

Figure 6a

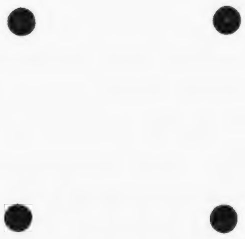

Figure 6b

Figure 6c

Figure 6d

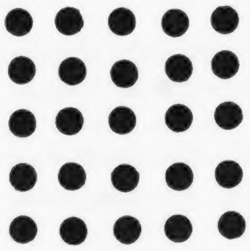

Figure 6e

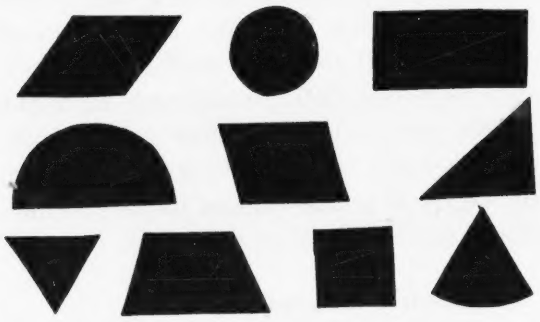

Figure 7

Figure 8a

Figure 8b

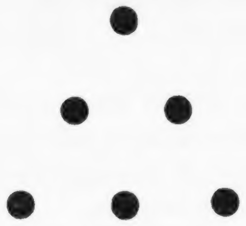

Figure 8c

Figure 8d

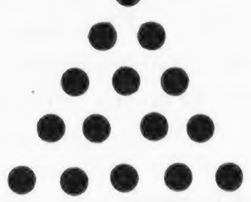

Figure 8e

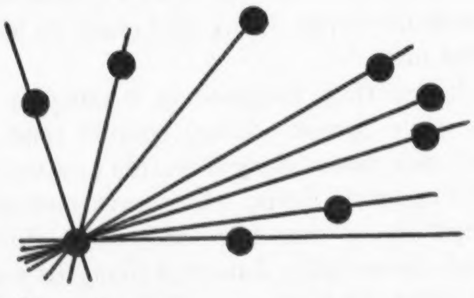

Figure 9

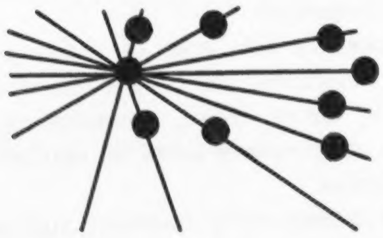

Figure 10

Later the dots represent nine. Then they help with the ancient idea that some numbers are squares: that nine corresponds to three threes. Furthermore, in

Figures 6*a–e*, the square arrays, one, four, nine, sixteen, twenty-five, etc., when ordered and compared via differences, encourage pupils to think about the odd numbers.

Square numbers:

—— 1 4 9 16 25 36 49 . . .

Differences between successive square numbers:

—— 3 5 7 9 11 13 . . .

Does zero belong in the blank before 1 in the top row?

For upper graders, finding a continuous path containing four line segments and connecting all the nine points will probably be a fascinating challenge. It might also be preparation for later work in simple topology, where, among other things, the study of paths—closed paths and not-closed paths—receives attention.

Geometric readiness

Examples such as the foregoing abound. Suppose that we look briefly at the pupils' readiness before we consider further instances.

We have hinted that readiness in geometry implies at least two other "nesses":

1. Preparedness, or adequate mathematical maturity to go on.
2. Willingness, or enough emotional security to go on.

Geometric preparedness begins at an early age. Tots in kindergarten enjoy plopping the cutout figures into their proper places (Fig. 7). Children quickly discriminate between right triangles and equilateral triangles, between squares and oblongs, and between trapezoids and parallelograms. Already these youngsters are shape conscious.

Success in this sort of activity leads young children on. Their handling of squares, cubes, disks, triangles, spheres, and so on, prepares them for further work with forms.

All too frequently, however, such activities terminate abruptly. This occurs because courses of study encourage the pupils to put away "childish things" and settle down to the stern business of memorizing facts and practicing operations with numbers. Since *perfection* in these worthy matters eludes most learners, the study of facts and operations flourishes while the study of forms languishes. Of course, lessons in the upper grades deal with areas and volumes, but computing with numbers and distinguishing between area and perimeter and between volume and surface have been known to monopolize the act.

Fortunately, the trend today points to geometry for the sake of geometry, rather than to geometry as further practice in calculating. In the elementary grades informal, or intuitive, studies get the emphasis. Drawing, counting, and measuring lead pupils to observing, inferring, and generalizing. Consciousness of forms continues to grow, and readiness for proofs in geometry also continues to grow.

Let us return momentarily to the tots. By handling wooden, paper, or plastic representations of geometric figures, children appreciate numerous ideas; among these are notions of square corners, straight edges, round edges like faces, roundness of disks, and roundness of balls. These children gain a degree of understanding to go on; they grow in geometric readiness.

But children gain in willingness too. The shapes, the fitting of objects into patterns, the matching, the comparing, and the counting all make children receptive to further activities. One quite ordinary first grader happened onto triangular-number arrangements, as in Figures 8*a–e*. Pupils do things, learn, and crave to learn some more.

So, as they progress in mathematical maturity (preparedness), pupils tend to seek new mathematical worlds to conquer (willingness). Thus, willingness and also preparedness stem from activities—things done successfully. Junior is likely to wonder what the next step will lead to. If, for example, thirty-six lines can be drawn through nine points such that no three points lie on a line (Fig. 9), then how many lines can be drawn if exactly three of the nine are on one line? (Fig. 10.) In the figures the joins of one point with each of the others constitute a hint; this is one way to begin the problems.

In sum, teachers seek to challenge pupils, not to frustrate. Sometimes a team approach (or working as a class on a perplexing problem) will prevent defeat. Whether pupils suffer more from frustration than from boredom, however, is moot.

Some simple arrangements

Besides patterns of points previously mentioned, we might look at a few other configurations. In Figure 11 a side of the square $ABCD$ measures 3, AE measures 2, and angle DEF measures 90°. In square $KLMN$, a side measures 2, and angle LKP has the same measure as angle BEF. The sections thus cut form two separate squares or one combined square—a kind of readiness for the Theorem of Pythagoras.

In Figure 12 $ABCD$ is a square with a side that measures 6, and $BEFG$ is a square with side 3. $AL = BH = CJ = DK = 1\frac{1}{2}$, and BN measures 3. M is the intersection of HK and JL. Square $MNOP$ has the same measure in square units as squares $ABCD$ and $BEFG$ combined.

Now suppose that we begin with other segments, half-squares and half-oblongs, for example. These, plus a few rectangles, semicircles, and quadrants, form a variety of designs. Figures 13–17 illustrate some of them.

Abundant triangles

If the pupils start with three equal segments and the question, "What can we do with the segments?", they can soon come up with an equilateral triangle (Fig. 18). By joining the mid-points of the sides of the triangle, they can produce four equilateral triangles (Fig. 19). By repeating the process successively for unshaded triangles, the pupils obtain Figures 20 and 21.

Or, the pupils may choose not to shade any of the component triangles and proceed to successive quartering of all the new triangles. One triangle yields four triangles, four yield sixteen, and sixteen yield sixty-four. Some pupils will move ahead and forge a fifth stage or even further proliferations of triangles. Theoretically, the fast workers need not grow weary of waiting for their slower classmates to finish a step. Endless steps await those who wish them, and the steps get harder.

Some pathological curves

If the pupils begin again with three equal segments, they can form another equilateral triangle (Fig. 22). By trisecting the sides, erecting equilateral triangles on the middle sections, and erasing the intersections of these four triangles, the pupils get Figure 23. Then in Figure 24 further trisections and outward points appear. Some pupils may wish to carry this procedure still further. Although the area of this snowflake-like curve clearly cannot exceed the surface of the page, its perimeter becomes infinitely large.

Similarly for other pathologic curves (Figs. 25–27), the pupils proceed from an equilateral triangle again. Here, however, the open mid-sections are spanned by equal segments that intersect inside instead of outside the original triangle. This gives an inverted snowflake pattern. Here too, the perimeter can be made infinitely large, even though the area will not exceed that of the drawing paper.

Figures 28, 29, and 30 show what results when pupils begin with a circle, divide it into six equal parts, and invert alternate arcs. This procedure, repeated, leads to another figure, the aesthetics of which may be doubtful. It is known as an inside-outside curve. It troubles almost everyone who seeks to determine its curvature.

Tiling patterns

As pupils soon learn when they begin to work with measures of angles, one full turn measures 360°. A further interesting investigation results when pupils face the question, "What flat figures will fit around

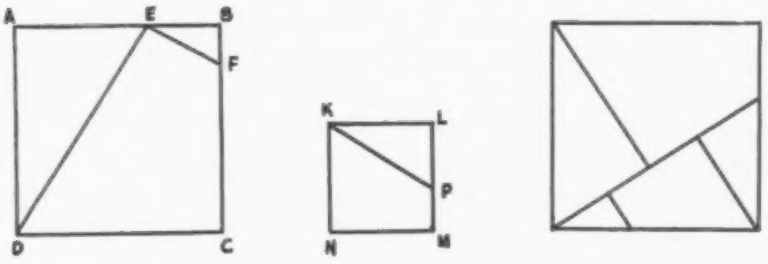
A
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Figure 11

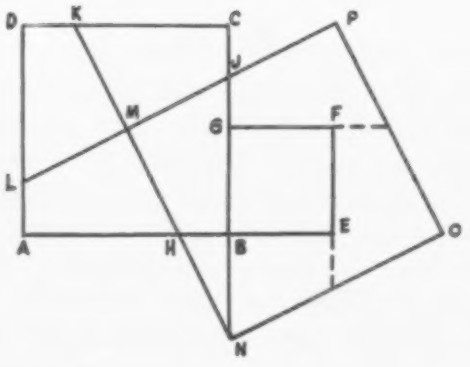
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Figure 12

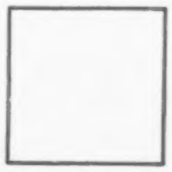


Figure 13

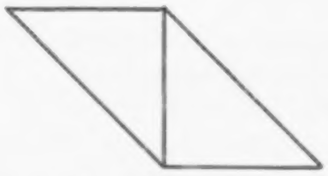


Figure 14

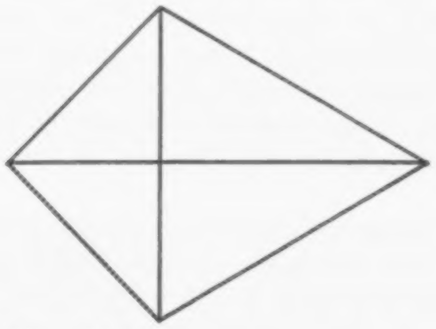


Figure 15

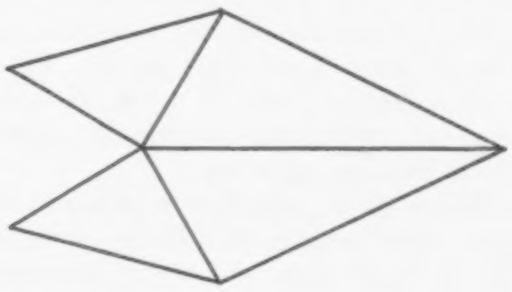


Figure 16

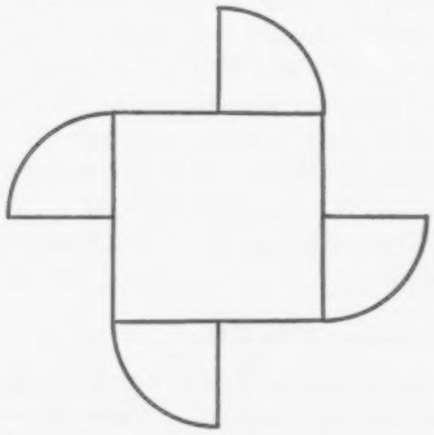


Figure 17

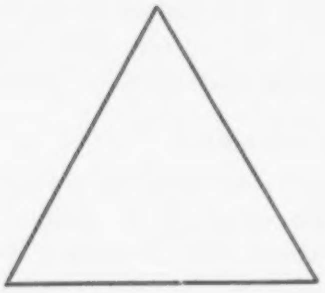


Figure 18

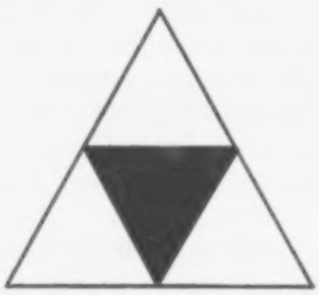


Figure 19

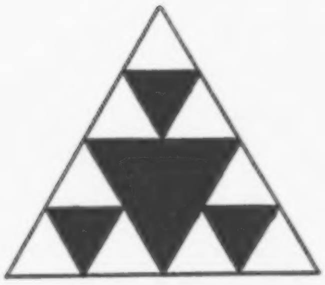


Figure 20

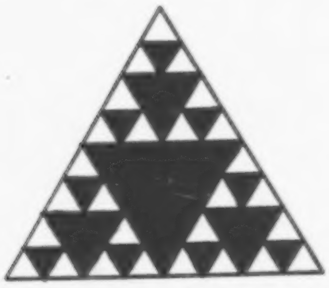


Figure 21

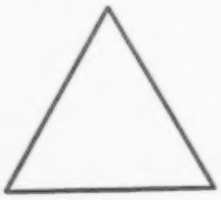
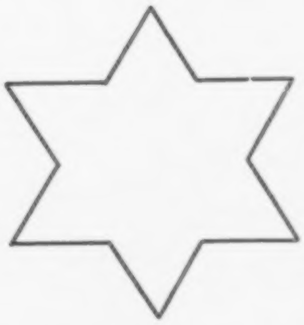


Figure 22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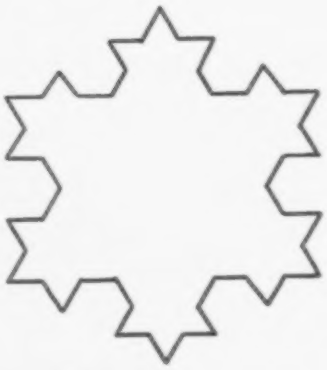
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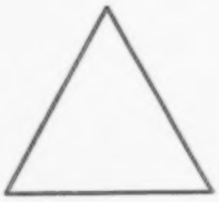
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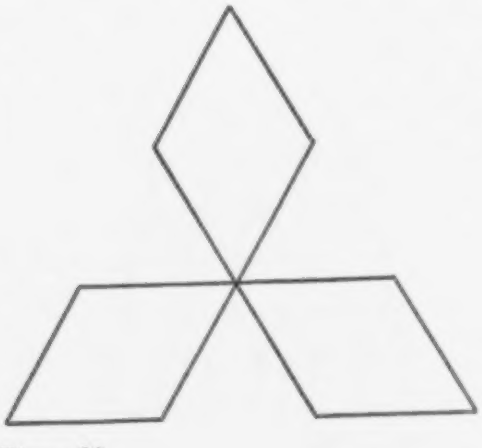
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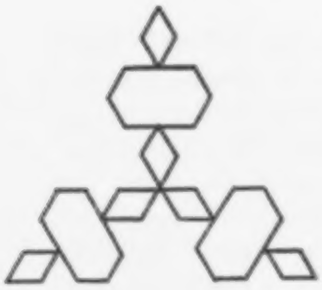
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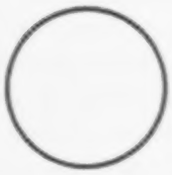
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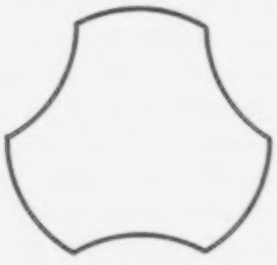
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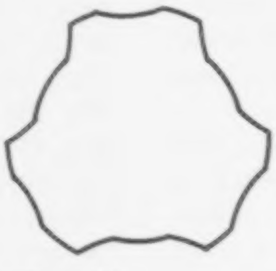
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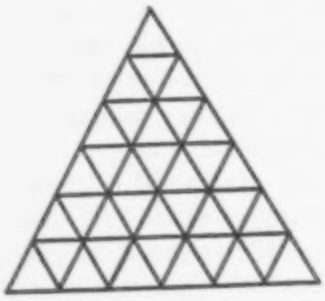
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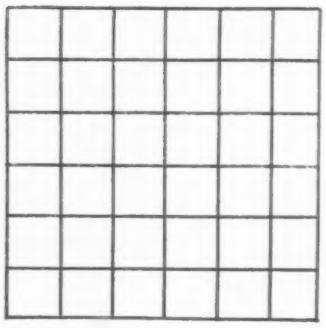

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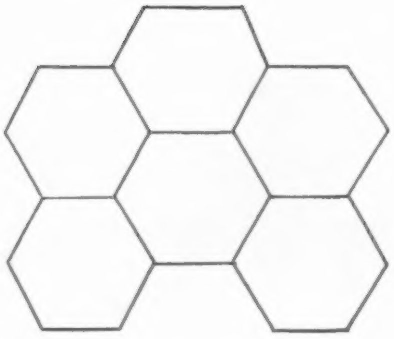

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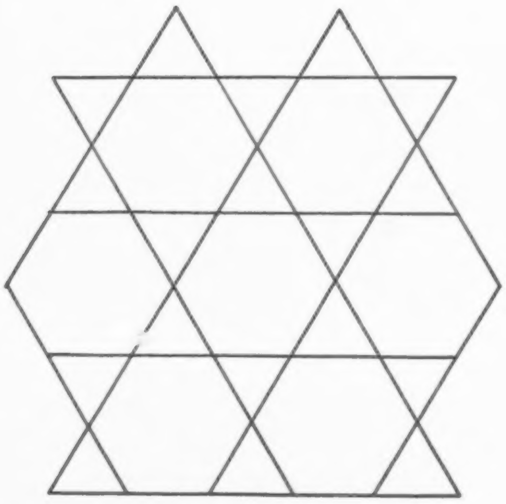

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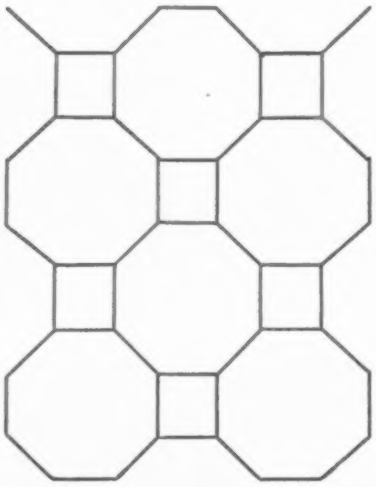

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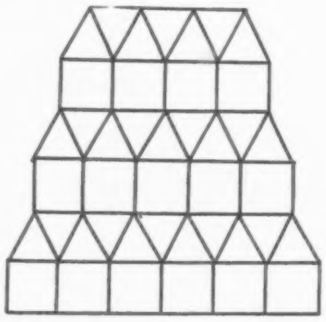

Figure 36

a point and fill in the flat surface?" Regular polygons seem to be needed, although all rectangles will suffice, but not all regular polygons will meet the requirement.

Considerable acquaintance with these polygons can result from such experimenting. How can we draw them? Later the pupils will study the straightedge-compass constructions for regular polygons, and still later they will study criteria of constructibility.

But strictly informal experimenting will reveal some combinations of polygons that, so to speak, cover the floor. Indeed, among sophisticates the whole subject of floor coverings, tilings, and mosaics bears the impressive name of "tessellation."

Six equilateral triangles (Fig. 31), four rectangles (Fig. 32), and three hexagons (Fig. 33) exhaust the possibilities of how many flat figures will fit around a point and fill in a flat surface. If, however, the pupils do not limit the problem to polygons of one single sort, then the following serve: two hexagons and two triangles (Fig. 34); two octagons and one square (Fig. 35); three triangles and two squares (Fig. 36); one hexagon, two squares, and one triangle (Fig. 37). Still other possibilities, not illustrated here, exist: one hexagon and four triangles; one dodecagon, one hexagon, and one square; two dodecagons and one triangle. Your pupils may want to try them.

It will occur to the pupils that these are possibilities when they experiment and construct the following table, referring to regular polygons:

Number of sides:

3 4 5 6 7 8 9 10 11 12

Measure of angles:

60 90 108 120 $128\frac{4}{7}$
135 140 144 $147\frac{3}{11}$ 150

From the increasing sizes of the angles, it appears that regular polygons having a still greater number of sides are not likely candidates.

Centroids

Locating a centroid, or a center of mass of a body, can become a thorny problem. Quite young children, on the other hand, can cut geometric forms from cardboard and readily locate lines and points of balance.

Regular figures balance rather easily on a knife-edge. The intersection of two such lines of balance determines a center of balance, or a center of mass. Equilateral triangles, squares, regular hexagons, reg-

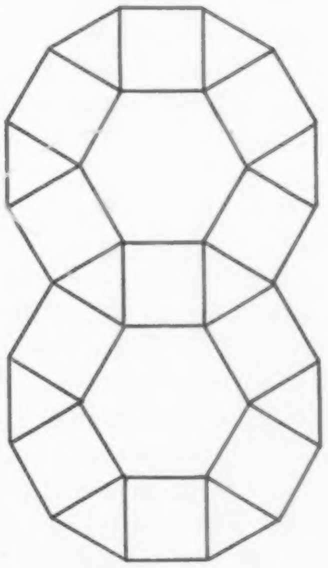

Figure 37

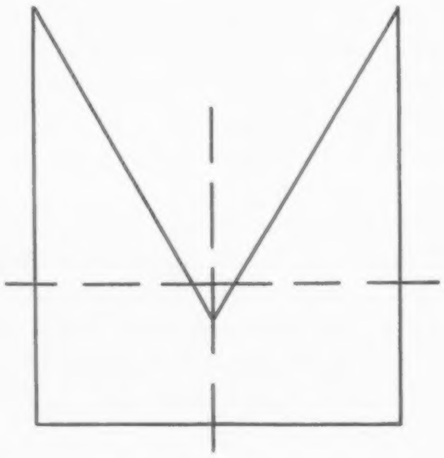

Figure 38

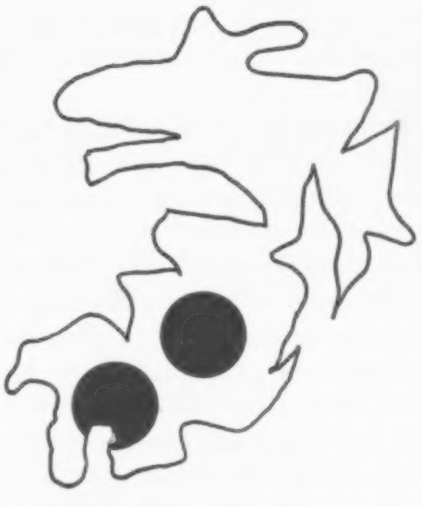

Figure 39

Figure 40

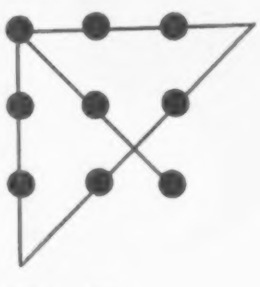

Figure 41

ular octagons, and circles illustrate these ideas.

When the figures depart from regularity, the knife-edge procedure may become more difficult, as in the case of a convex polygon. There the centroid lies outside the figure (Fig. 38). This might become the germ of the idea of a centroid for a *system* of bodies, which may interest the future physicists and astronomers in your classes.

Simple and nonsimple

Finding the centroid of an involved, yet technically simple, curve could provide a difficult problem. A cutout again provides an intuitive approach (Fig. 39). Incidentally, when is a curve simple? If the left circle were removed from Figure 39, the curve would remain simple. But if the right circle were removed, the figure would become nonsimple.

Figure 40 is another puzzler. It has two outsides, one of which might seem at first to be inside. When is an outside inside?

In case you wondered, here is one solution to the path problem, referred to earlier, for nine points arranged as three threes (Fig. 41).

Yin-yang

Rooted in antiquity, especially in venerable Chinese philosophy, is the symbol represented in Figure 42. From *yang*, literally the south or sunny side of a hill, the unshaded portion of the design represents the bright, good, positive, male principle in Chinese dualism. *Yin*, on the contrary, symbolizes the dark, evil, negative, and female. The shaded part stands for *yin*.

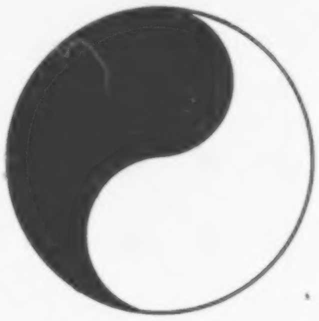

Figure 42

Regardless of how pupils and their teachers look on such mystical matters, the emblem appears to be easy to construct and to describe geometrically.

Summary

Suppose, now, that we summarize. From the kindergarten on the concepts, rules, and operations of arithmetic and algebra dominate pupils' experiences in mathematics. Few deny it, and if the teaching has been good, still fewer bemoan it.

New occasions, however, teach new duties. Mathematics grows apace; mathematics education accelerates its search. With reason we urge teachers to learn new mathematics and teach new courses. We see merit in helping young children to gain acquaintance with good mathematics early. We note with pleasure the improvements in textbooks of elementary-school mathematics.

In our zeal, however, we run the risk of letting abstractions get out of hand. This we disapprove. The motto "be abstract" should be left to the habitants of Greenwich Village. The race, including those of us who urge mathematics reforms, learned mathematics through practical needs and real problems. The abstractions, the generalizations, and the deductions followed the investigations, the approximations, and the corrections.

Surely, therefore, we should not deny young children the opportunity to explore and learn.

A proper study for all children is geometry—the geometry of form. Here pupils perceive, compare, measure, and generalize. Here they sharpen intuition without plunging too far into abstractions. Above all, children see values in what they do. If we can encourage pupils to discover for themselves some principles in the science of space, then they will bring into their geometry classes a usable store of information about the Euclidean plane. They might also have a good start on three-space.

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